Analytic Evaluation of QED Corrections in One-Electron Ions

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Outline

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Motivation

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- Theory of Everything

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- Testing Current Fundamental Theories

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$\Longrightarrow$ High-Precision Theory!

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Quantum Electrodynamics (QED)

[1] https://www.mpi-hd.mpg.de/mpi/fileadmin/bilder/Progress_Reports/2017-19/2QuantumDynamics.pdf

Quantum Electrodynamics (QED)

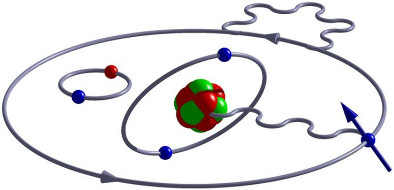

Figure: Scheme of the QED contributions to the electronic structure of highly charged ions.[1]

$$\mathcal{L}_{\mathrm{QED}} = \bar{\psi}\left[\gamma^{\mu}\left(i\hbar c\partial_{\mu} - eA_{\mu}\right) - m_e c^2\right]\psi - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}\,. \qquad (1)$$

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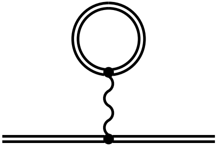
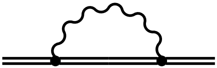
One-Loop Energy Corrections

Vacuum Polarization:



Self-Energy Correction:



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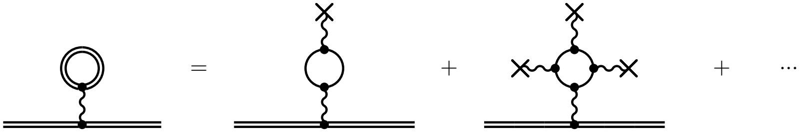

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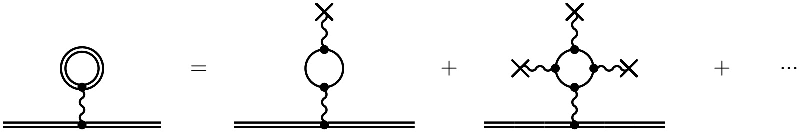
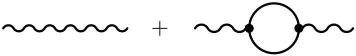


Modification of the Photon Propagator:

$iD'_{\mu\nu}(k) =$



$$= iD_{\mu\nu}(k) + iD_{\mu\lambda}(k)\frac{i\Pi^{\lambda\sigma}(k)}{4\pi}iD_{\sigma\nu}(k)$$

Inspired by [9]

One-Loop g Factor Corrections

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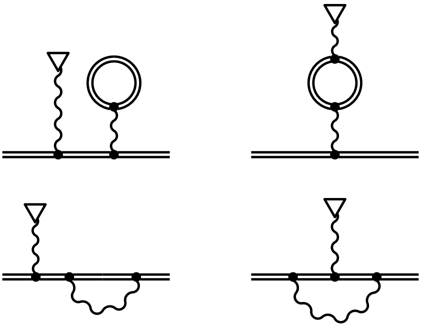

Figure: Feynman diagrams representing the first-order radiative corrections to the g factor of the bound electron.

g Factor

The g Factor is given by [18]:

$$g = -\frac{\kappa}{2j(j+1)} \left(1 - 2\kappa \frac{\partial E_{n\kappa}}{\partial m_e}\right) , \tag{2}$$

if the potential $V(r)$ does not depend on the electron mass m_e.

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Small Perturbation of the potential $\delta V(r)$ leads to

$$\Delta g = -\frac{\kappa^2}{j(j+1)m_e} \left\langle r \frac{\partial \delta V(r)}{\partial r} \right\rangle . \tag{3}$$

Outline

Leptonic Vacuum Polarization

Leptonic Uehling Potential [9]:

$$\delta V(r) = \frac{\alpha}{\pi} \int_0^1 dv \, \frac{v^2 \left(1 - v^2/3\right)}{1 - v^2} \left(-\frac{Z\alpha}{r} e^{-2m_l r/\sqrt{1-v^2}} \right), \quad (4)$$

where m_l is the mass of the virtual particle in the fermionic loop.

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Energy Shift of the $1s$ State:

$$\Delta E_{1s}^{\text{lept. VP}} = \langle \delta V(r) \rangle_{1s} = \int_0^\infty dr \; \left(G_{1s}^2(r) + F_{1s}^2(r) \right) \delta V(r). \quad (5)$$

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g Factor Shift of the 1s State:

$$\Delta g_{1s}^{\text{lept. VP}} = -\frac{4}{3m_e} \left\langle r \frac{\partial \delta V(r)}{\partial r} \right\rangle_{1s}. \qquad (6)$$

4.1. Energy Shift

We now perform the calculations for the energy shift of the $1s$ state in hydrogen-like atoms due to the leptonic Uehling potential. When $|\psi_{1s}\rangle$ denotes the bound electron wave function of the ground state in a point-like Coulomb potential, the energy shift in first-order perturbation theory is given by:

$$
\begin{aligned}
\Delta E_{1s}^{\mathrm{VP}} &= \langle\psi_{1s}|\delta V(r)|\psi_{1s}\rangle = \int_0^\infty dr\, \left(G^2(r)+F^2(r)\right)\delta V(r) \\
&= \frac{2\lambda}{\Gamma(2\gamma+1)}\int_0^\infty dr\,(2\lambda r)^{2\gamma}e^{-2\lambda r}\left(-\frac{Z\alpha^2}{\pi r}\int_0^1 dv\,\frac{v^2\left(1-v^2/3\right)}{1-v^2}e^{-2m_l r/\sqrt{1-v^2}}\right) \\
&= -\frac{2\lambda Z\alpha^2}{\Gamma(2\gamma+1)\pi}\int_0^1 dv\,\frac{v^2\left(1-v^2/3\right)}{1-v^2}\int_0^\infty dr\,\frac{(2\lambda r)^{2\gamma}}{r}e^{-2r\left(\lambda+\frac{m_l}{\sqrt{1-v^2}}\right)}. \qquad (4.4)
\end{aligned}
$$

Now, substitute $u=r^{2\gamma}$, $du = 2\gamma r^{2\gamma-1}dr$,
and then $t = \left[2\left(\lambda+\frac{m_l}{\sqrt{1-v^2}}\right)\right]^{2\gamma}u$, $dt = \left[2\left(\lambda+\frac{m_l}{\sqrt{1-v^2}}\right)\right]^{2\gamma}du$:

$$
\begin{aligned}
\Delta E_{1s}^{VP} &= -\frac{2\lambda Z\alpha^2}{\Gamma(2\gamma+1)\pi}\int_0^1 dv\,\frac{v^2\left(1-v^2/3\right)}{1-v^2}\int_0^\infty \frac{du}{2\gamma}(2\lambda)^{2\gamma}e^{-2u^{1/2\gamma}\left(\lambda+\frac{m_l}{\sqrt{1-v^2}}\right)} \\
&= -\frac{2\lambda Z\alpha^2}{\Gamma(2\gamma+1)\pi}\int_0^1 dv\,\frac{v^2\left(1-v^2/3\right)}{1-v^2}\,\frac{(2\lambda)^{2\gamma}}{2\gamma\left[2\left(\lambda+\frac{m_l}{\sqrt{1-v^2}}\right)\right]^{2\gamma}}\underbrace{\int_0^\infty dt\, e^{-t^{1/2\gamma}}}_{=2\gamma\Gamma(2\gamma)} \\
&= -\frac{Z\alpha^2\lambda^{2\gamma+1}}{\gamma\pi}\int_0^1 dv\,\frac{v^2\left(1-v^2/3\right)\left(1-v^2\right)^{\gamma-1}}{\left[\lambda\sqrt{1-v^2}+m_l\right]^{2\gamma}} \\
&= -\frac{Z\alpha^2(sZ\alpha)^{2\gamma}\lambda}{\gamma\pi}\int_0^1 dv\,\frac{v^2\left(1-v^2/3\right)\left(1-v^2\right)^{\gamma-1}}{\left[1+sZ\alpha\sqrt{1-v^2}\right]^{2\gamma}}, \qquad (4.5)
\end{aligned}
$$

where we used the relation $\Gamma(x+1)=x\Gamma(x)$ and that $\lambda/m_l = sZ\alpha$, where $s=m_e/m_l$ is the ratio of the electron and the loop particle masses.

This integral can now be solved analytically with the base integral given in [28]:

$$
\begin{aligned}
I_{abc} &= \int_0^1 dy\,\frac{\left(1-y^2\right)^{a-1/2}}{y^{b-1}}\left(\frac{sZ\alpha y}{1+sZ\alpha y}\right)^{c-2\epsilon} \\
&= \frac{1}{2}(sZ\alpha)^{c-2\epsilon}B\left(a+\frac{1}{2},1-\frac{b-c}{2}-\epsilon\right) \\
&\times\ {}_3F_2\left(\frac{c}{2}-\epsilon,\frac{c+1}{2}-\epsilon,1-\frac{b-c}{2}-\epsilon;\frac{1}{2},a+\frac{3-b+c}{2}-\epsilon;(sZ\alpha)^2\right) \\
&- \frac{c-2\epsilon}{2}(sZ\alpha)^{c+1-2\epsilon}B\left(a+\frac{1}{2},\frac{3-b+c}{2}-\epsilon\right) \\
&\times\ {}_3F_2\left(\frac{c}{2}+1-\epsilon,\frac{c+1}{2}-\epsilon,\frac{3-b+c}{2}-\epsilon;\frac{3}{2},a+2-\frac{b-c}{2}-\epsilon;(sZ\alpha)^2\right). \qquad (4.6)
\end{aligned}
$$

Here, $\epsilon = 1-\gamma$, and $B(x,y)$ is the beta function defined in Appendix A.4. Performing the substitution $y=\sqrt{1-v^2}$, $dy = \frac{-vdv}{\sqrt{1-v^2}}$, one obtains finally:

$$
\begin{aligned}
\Delta E_{1s}^{\mathrm{VP}} &= -\frac{Z\alpha^2(sZ\alpha)^{2\gamma}\lambda}{\gamma\pi}\left[\int_0^1\frac{ydy}{\sqrt{1-y^2}}\frac{\left(1-y^2\right)y^{2\gamma-2}}{\left[1+sZ\alpha y\right]^{2\gamma}}-\frac{1}{3}\int_0^1\frac{ydy}{\sqrt{1-y^2}}\frac{\left(1-y^2\right)^2y^{2\gamma-2}}{\left[1+sZ\alpha y\right]^{2\gamma}}\right] \\
&= -\frac{Z\alpha^2\lambda}{\gamma\pi}\left[\int_0^1 dy\,\frac{\left(1-y^2\right)^{\frac{1}{2}}}{y}\left(\frac{sZ\alpha y}{1+sZ\alpha y}\right)^{2\gamma}-\int_0^1 dy\,\frac{\left(1-y^2\right)^{\frac{3}{2}}}{y}\left(\frac{sZ\alpha y}{1+sZ\alpha y}\right)^{2\gamma}\right] \\
&= -\frac{Z\alpha^2\lambda}{\gamma\pi}\left[I_{122}-\frac{1}{3}I_{222}\right]. \qquad (4.7)
\end{aligned}
$$

This result was already derived in [27] and reference therein. The difference is that in [27], the result was derived for the electron as loop-particle and electrons or muons as bound particles. Here, we considered the electron as bound particle and arbitrary leptons as loop-particles.

To investigate this all-order in $Z\alpha$ result, we may expand (4.7) in a series for small $Z\alpha$ to get a simple and handy formula for the correction due to the leptonic Uehling potential. Using the computer algebra software *Mathematica* [18], we find for the leading order Taylor expansion:

$$
\begin{aligned}
\Delta E_{1s}^{\mathrm{VP}} &= \frac{\alpha m_e}{\pi}\left[-\frac{4s^2(Z\alpha)^4}{15}+\frac{5\pi s^3(Z\alpha)^5}{48}\right. \\
&+\left(\frac{4s^2}{15}\ln(2sZ\alpha)-\frac{107s^2}{225}-\frac{12s^4}{35}\right)(Z\alpha)^6 \\
&+\left.\left(-\frac{5\pi s^3}{48}\ln\left(\frac{sZ\alpha}{2}\right)-\frac{17\pi s^3}{576}+\frac{7\pi s^5}{64}\right)(Z\alpha)^7+\mathcal{O}\left((Z\alpha)^8\right)\right]. \qquad (4.8)
\end{aligned}
$$

Therefore, the energy shift correction for leptonic vacuum polarization is of the leading order $(Z\alpha)^4$. The first term of expression (4.8) coincides with the result of the simplest approximation, namely, using nonrelativistic wave functions and the low-momentum approximation of the polarization function [13, 16, 24]. Since this leads to a Dirac delta function, we will call it the δ-potential approximation.

With formula (4.7), one can calculate the energy shift contribution for any leptonic vacuum loop.[2] As equation (4.7) or (4.8) show, the heavier the loop particle is, the smaller is the measureable effect. After the e^-e^+-loop, the next important contribution is due to the $\mu^-\mu^+$-loop, also called muonic vacuum polarization, where a muon and anti-muon pair is produced in the loop. The effect of muonic vacuum polarization is around $s^2 \approx 1/207^2 \approx 2\cdot 10^{-5}$ times smaller than that of the electronic vacuum polarization.

Figure 4.1 shows the absolute value of the muonic vacuum polarization contribution to the energy shift for different charge numbers Z, calculated with the exact formula (4.7) and its approximation (4.8) up to 5th order. It should be mentioned that the Uehling potential is attractive and results in a stronger binding. Thus, the corresponding energy shift is negative. The correction for ions with charge numbers $Z \leq 8$ is so small that its value is not visible in this diagram.

[2] As mentioned in [28], this formula can be applied to regular, electronic atoms or ions considered in this work, or to muonic atoms by replacing the electron mass m_e with the muon mass m_μ.

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Energy Shift for a point-like Nucleus:

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g Factor Shift for a point-like Nucleus:

$$\Delta g_{1s}^{\text{lept. VP}} = -\frac{8\alpha(Z\alpha)}{3\pi s}\left[I_{133} - \frac{1}{3}I_{233} + \frac{Z\alpha s}{2\gamma}\left(I_{122} - \frac{1}{3}I_{222}\right)\right]. \tag{8}$$

Hadronic Vacuum Polarization

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Real Part of the Hadronic Polarization Function [4]:

$$\mathrm{Re}\left[\Pi_{\mathrm{had}}(q^2)\right] = A_i + B_i \ln(1 + C_i q^2), \tag{9}$$

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The parameters are given by

i	Region	Range [GeV]	A_i	B_i	C_i [GeV^{-2}]
1	0 - k_1	0.0 - 0.7	0.0	0.0023092	3.9925370
2	k_1 - k_2	0.7 - 2.0	0.0	0.0022333	4.2191779
3	k_2 - k_3	2.0 - 4.0	0.0	0.0024402	3.2496684
4	k_3 - k_4	4.0 - 10.0	0.0	0.0027340	2.0995092
5	k_4 - k_5	10.0 - m_Z	0.0010485	0.0029431	1.0
6	k_5 - k_6	m_Z - 10^4	0.0012234	0.0029237	1.0
7	k_5 - k_6	10^4 - 10^5	0.0016894	0.0028984	1.0

Hadronic Uehling Potential

Numerical Hadronic Uehling Potential for a point-like Nucleus:

$$\delta V^{\text{had. VP}}_{\text{numerical}}(r) = -\frac{2Z\alpha}{\pi} \sum_{k=1}^{7} \int_{k_{i-1}}^{k_i} dq \, \frac{\sin(qr)}{qr} \left[A_i + B_i \ln(1 + C_i q^2) \right] . \tag{10}$$

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Analytical Hadronic Uehling Potential for a point-like Nucleus:

$$\begin{aligned} \delta V^{\text{had. VP}}_{\text{analytical}}(r) &= -\frac{2Z\alpha}{\pi} \int_0^{\infty} dq \, \frac{\sin(qr)}{qr} \left[A_1 + B_1 \ln(1 + C_1 q^2)\right] \\ &= -\frac{2Z\alpha}{r} B_1 \, \mathrm{E}_1\left(\frac{r}{\sqrt{C_1}}\right) . \end{aligned} \tag{11}$$

Hadronic Uehling Potential

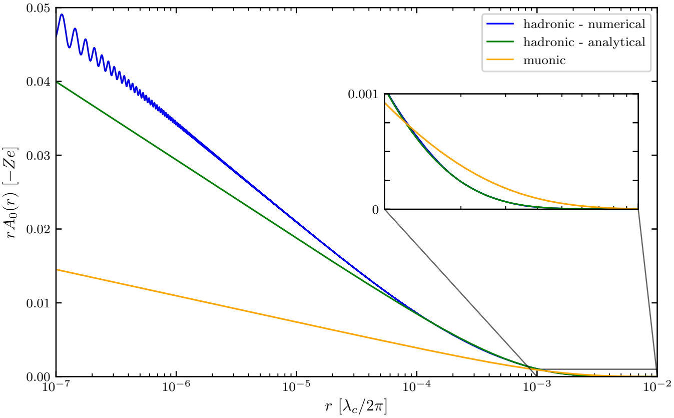

Figure: Comparison of leptonic and hadronic Uehling potential.

Hadronic Energy Shift

Hadronic Energy Shift

Energy Shift of the 1s State for a point-like Nucleus:

$$\Delta E_{1s}^{\text{had. VP}} = -\frac{Z\alpha\lambda(2\lambda\sqrt{C_1})^{2\gamma}B_1}{\gamma^2}\ {}_2F_1\left(2\gamma, 2\gamma; 1+2\gamma; -2\lambda\sqrt{C_1}\right). \tag{12}$$

Hadronic Energy Shift

Energy Shift of the $1s$ State for a point-like Nucleus:

$$\Delta E_{1s}^{\text{had. VP}} = -\frac{Z\alpha\lambda(2\lambda\sqrt{C_1})^{2\gamma} B_1}{\gamma^2} \, {}_2F_1\left(2\gamma, 2\gamma; 1+2\gamma; -2\lambda\sqrt{C_1}\right). \tag{12}$$

$Z\alpha$ Expansion:

$$\begin{aligned}\Delta E_{1s}^{\text{had. VP}} \approx & -4B_1 C_1 m_e^3 (Z\alpha)^4 + \frac{32 B_1 C_1^{3/2} m_e^4 (Z\alpha)^5}{3} \\ & -4B_1 C_1 m_e^3 (Z\alpha)^6 \left[1 + 6C_1 m_e^2 - \ln(2Z\alpha\sqrt{C_1} m_e)\right].\end{aligned} \tag{13}$$

Hadronic Energy Shift

Z	$\Delta E^{\text{point}}_{\text{analytical}}$ [eV]	$\Delta E^{\text{point}}_{\text{numerical}}$ [eV]	$\Delta E^{\text{finite size}}_{\text{exact}}$ [eV]
1	$-1.3963 \cdot 10^{-11}$	$-1.39(33) \cdot 10^{-11}$	$-1.391(4) \cdot 10^{-11}$
14	$-5.9178 \cdot 10^{-7}$	$-5.90(18) \cdot 10^{-7}$	$-5.756(1) \cdot 10^{-7}$
20	$-2.7133 \cdot 10^{-6}$	$-2.71(5) \cdot 10^{-6}$	$-2.5596(3) \cdot 10^{-6}$
70	$-3.1090 \cdot 10^{-3}$	$-3.109(4) \cdot 10^{-3}$	$-1.248(1) \cdot 10^{-3}$
82	$-1.4128 \cdot 10^{-2}$	$-1.413(1) \cdot 10^{-2}$	$-3.693(4) \cdot 10^{-3}$

Table: Energy shifts for hadronic VP.

Hadronic g Factor Shift

Hadronic g Factor Shift

g Factor Shift of the $1s$ State for a point-like Nucleus:

$$\Delta g_{1s}^{\text{had. VP}} = -\frac{8B_1(Z\alpha)^2(2\lambda\sqrt{C_1})^{2\gamma}}{3\gamma(1+2\lambda\sqrt{C_1})^{2\gamma}} + \frac{4}{3m_e}\Delta E_{\text{approx}}. \tag{14}$$

Hadronic g Factor Shift

g Factor Shift of the 1s State for a point-like Nucleus:

$$\Delta g_{1s}^{\text{had. VP}} = -\frac{8B_1(Z\alpha)^2(2\lambda\sqrt{C_1})^{2\gamma}}{3\gamma(1+2\lambda\sqrt{C_1})^{2\gamma}} + \frac{4}{3m_e}\Delta E_{\text{approx}}. \qquad (14)$$

$Z\alpha$ Expansion:

$$\Delta g_{1s}^{\text{had. VP}} \approx -16B_1C_1m_e^2(Z\alpha)^4 + \frac{512B_1C_1^{3/2}m_e^3(Z\alpha)^5}{9} - \frac{16B_1C_1m_e^2(Z\alpha)^6}{3}\left[2+30C_1m_e^2-3\ln(2m_eZ\alpha\sqrt{C_1})\right]. \qquad (15)$$

Hadronic g Factor Shift

Z	$\Delta g^{\text{point}}_{\text{analytical}}$	$\Delta g^{\text{point}}_{\text{numerical}}$	$\Delta g^{\text{finite size}}_{\text{approx}}$
1	$-1.0929 \cdot 10^{-16}$	$-1.09(9) \cdot 10^{-16}$	$-1.09(2) \cdot 10^{-16}$
14	$-4.6157 \cdot 10^{-12}$	$-4.61(5) \cdot 10^{-12}$	$-4.49(1) \cdot 10^{-12}$
20	$-2.1085 \cdot 10^{-11}$	$-2.11(2) \cdot 10^{-11}$	$-1.99(1) \cdot 10^{-11}$
70	$-2.2051 \cdot 10^{-8}$	$-2.205(1) \cdot 10^{-8}$	$-8.86(1) \cdot 10^{-9}$
82	$-9.5886 \cdot 10^{-8}$	$-9.589(3) \cdot 10^{-8}$	$-2.51(1) \cdot 10^{-8}$

Table: g factor shifts for hadronic VP.

Outline

Self-Energy Correction

$$\Delta E^{\text{SE}} = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \left\langle \psi \left| \gamma^\mu \frac{1}{\not{p} - \not{k} - m_e - \gamma^0 V} \gamma_\mu \right| \psi \right\rangle . \tag{16}$$

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- Exact Coulomb-Dirac Propagator only known in Coordinate Space
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$\Longrightarrow$ Approximation!

Dimensional Regularization

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This Regularization Technique yields the well-known Result [9]

$$\Delta E_{1s}^{\mathrm{SE}} = \frac{4\alpha}{3\pi}(Z\alpha)^4 m_e \left[\frac{5}{6} - 2\ln(Z\alpha) - \ln k_0\right], \qquad (17)$$

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where the Bethe logarithm $\ln k_0$ is defined as

$$\frac{(Z\alpha)^4 m_e}{n^3} \ln k_0 = \frac{1}{2m_e^2} \left\langle \boldsymbol{p}\,(H - E)\ln\left[\frac{2(H-E)}{m_e(Z\alpha)^2}\right]\boldsymbol{p}\right\rangle . \qquad (18)$$

Bethe Logarithm

Integral representation of the Bethe logarithm [13]:

$$\ln k_0(n) = -\frac{3}{4}\,PV\int_0^1 dt\,\frac{1}{t^3}\left(\frac{t^2-1}{nt^2}P_{nd}(t)+\frac{2}{3n}-\frac{8t^2}{3}\right) - 2\ln(n), \tag{19}$$

PV denotes the principal value and $P_{nd}(t)$ is the non-relativistic dipol matrix element:

$$P_{nd} = \frac{1}{3m_e}\left\langle \phi_{nlm}\left|\boldsymbol{p}\frac{1}{H-(E-\omega)}\boldsymbol{p}\right|\phi_{nlm}\right\rangle . \tag{20}$$

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$\Longrightarrow$ Plenty of Room for Analytical Work!

Thank you and stay healthy!

References I

Abramowitz, M. ; Stegun, I. A.:
Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables.
New York : Dover, 1964

Bethe, H. A.:
The Electromagnetic Shift of Energy Levels.
In: *Physical Review* 72 (1947), Nr. 339, S. 339–341.
http://dx.doi.org/10.1103/PhysRev.72.339. –
DOI 10.1103/PhysRev.72.339. –
ISSN 03702693

References II

Breidenbach, J.S.:
Hadronic vacuum polarization in atoms.
2018

Burkhardt, H. ; Pietrzyk, B.:
Update of the hadronic contribution to the QED vacuum polarization.
In: *Physics Letters B* 513 (2001), S. 46–52.
http://dx.doi.org/10.1016/S0370-2693(01)00393-8. –
DOI 10.1016/S0370–2693(01)00393–8. –
ISSN 03702693

References III

Drake, G. W. F. ; Swainson, A.:
Bethe logarithms for hydrogen up to $n = 20$, and approximations for two-electron atoms.
In: *Physical Review A* 41 (1990), Nr. 1243.
http://dx.doi.org/10.1103/PhysRevA.41.1243. –
DOI 10.1103/PhysRevA.41.1243

Fließbach, T.:
Quantenmechanik.
6. Auflage.
Berlin : Springer Spektrum, 2018. –
ISBN 9783662580301

References IV

Friar, J. L. ; Martorell, J. ; Sprung, D. W.:
Hadronic Vacuum Polarization and the Lamb Shift.
In: *Physical Review A* 59 (1999), Nr. 4061, S. 4061–4063.
http://dx.doi.org/10.1103/PhysRevA.59.4061. –
DOI 10.1103/PhysRevA.59.4061. –
ISSN 10941622

Greiner, W.:
Relativistic Quantum Mechanics - Wave Equations.
Third Edition.
Berlin, Heidelberg, New York : Springer-Verlag, 2000. –
ISBN 3–540–67457–8

References V

Greiner, W. ; Reinhardt, J.:
Quantum Electrodynamics.
Fourth Edition.
Berlin, Heidelberg : Springer-Verlag, 2009. –
ISBN 9788578110796

Harman, Z. ; Quint, W.:
QED - Theory and Modern Experiments.
Lecture Notes.
Heidelberg University, 2019. –
Available at https://www.mpi-hd.mpg.de/personalhomes/harman/qed2019.html

References VI

Inc., Wolfram R.:
Mathematica.
Version 12.0.
Champaign, Illinois : Wolfram Research, Inc., 2019

Itzykson, C. ; Zuber, J. B.:
Quantum Field Theory.
New York : McGraw-Hill, 1980. –
ISBN 9780486445687

Jentschura, U. D.:
Theory of the Lamb Shift in Hydrogenlike Systems.
2003

References VII

Kinoshita, T.:
Quantum Electrodynamics.
World Scientific, 1990

Kusch, P. ; Foley, H. M.:
Precision Measurement of the Ratio of the Atomic 'g Values' in the $^2P_{3/2}$ and $^2P_{1/2}$ States of Gallium.
In: *Physical Review* 72 (1947), Nr. 1256.
http://dx.doi.org/10.1103/PhysRev.72.1256.2. – DOI 10.1103/PhysRev.72.1256.2

References VIII

Lamb, W. E. ; Retherford, R. C.:
Fine Structure of the Hydrogen Atom by a Microwave Method.
In: *Physical Review* 72 (1947), Nr. 241, S. 241–243.
http://dx.doi.org/10.1103/PhysRev.72.241. –
DOI 10.1103/PhysRev.72.241

Landau, L. D. ; Lifshitz, E. M.:
Quantenelektrodynamik.
Europa-Lehrmittel, 1991. –
ISBN 978–3–8085–5632–0

Michel, N.:
Precision Physics of the Bound Electron g-factor.
2015

References IX

Peskin, M. E. ; Schroeder, D. V.:
An Introduction to Quantum Field Theory.
Reading, USA : Addison-Wesley, 1995. –
ISBN 9780201503975

Sikora, B.:
Quantum field theory of the g-factor of bound systems.
2018