Hadronic Vacuum Polarization Corrections in Highly Charged Ions

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Abstract

The hadronic vacuum polarization corrections to atomic energy levels and the g factor of a bound electron are investigated theoretically. A parametric hadronic polarization function obtained from experimental cross sections of e^-e^+ annihilation into hadrons is applied to derive an effective relativistic Uehling potential. The energy and g factor corrections are calculated for low-lying hydrogenic levels using analytical Dirac-Coulomb wave functions, as well as with bound wave functions accounting for the finite nuclear size. Closed formulas for the hadronic Uehling potential of an extended nucleus as well as for the relativistic shifts in case of a point-like nucleus are derived. These results are compared to analytical formulas from non-relativistic theory and numerical values for extended nuclei.

Vacuum polarization

In quantum electrodynamics (QED), the electromagnetic interaction is described by the exchange of virtual photons between charged particles. The presence of virtual particle-antiparticle pairs, such as e^-e^+ and $\mu^-\mu^+$, further gives rise to a perturbation to the electron-nucleus interaction and can be detected by high-precision experiments.

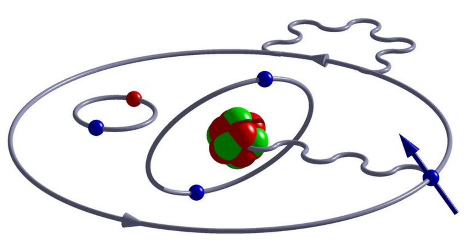

Figure 1. QED contributions to the electronic structure of highly charged ions [1].

The vacuum polarization effect for leptons in the vacuum loop is well known and can be described by perturbing potentials. The leading-order term is called the Uehling potential. In this work, we discuss the Uehling contribution arising from hadronic pair creation.

Hadronic energy correction

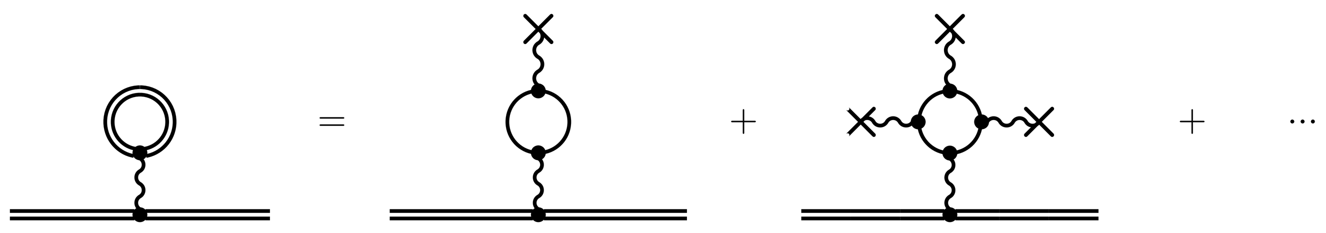

Figure 3. Vacuum polarization contribution to the bound-electron energy levels [4].

Analytical results

In first-order perturbation theory, the energy correction ΔE due to a small perturbation potential $\delta V(r)$ is given by

$$\Delta E = \langle \delta V(r) \rangle \, .$$

The non-relativistic approximation for a point-like nucleus can be obtained with

$$\delta V^{\rm had.}_{\rm non-rel.,point}(\boldsymbol{x}) = -4\pi Z\alpha B_1 C_1 \delta^{(3)}(\boldsymbol{x}) \, .$$

The full relativistic calculation for a point-like nucleus yields

$$\Delta E^{\rm had.}_{\rm rel.,point}(1s) = -\frac{Z\alpha\lambda(2\lambda\sqrt{C_1})^{2\gamma}B_1}{\gamma^2}\, {}_2F_1\left(2\gamma, 2\gamma; 1+2\gamma; -2\lambda\sqrt{C_1}\right)$$
$$= -4B_1C_1m_e^3(Z\alpha)^4 + \frac{32B_1C_1^{3/2}m_e^4(Z\alpha)^5}{3} + \ldots \, ,$$

where m_e is the electron mass, $\gamma = \sqrt{1-(Z\alpha)^2}$ and $\lambda = Z\alpha m_e$.

Numerical results

The hadronic vacuum polarization contribution is found to be around 67 % of the muonic vacuum polarization contribution. Numerical values for point-like and extended (fns) nuclei for several low-lying hydrogenic levels were calculated. Tab. 1 shows some results.

Z	$\Delta E^{\rm had.}_{\rm non-rel.,point}(1s)$ [eV]	$\Delta E^{\rm had.}_{\rm rel.,point}(1s)$ [eV]	$\Delta E^{\rm had.}_{\rm rel.,fns}(1s)$ [eV]
1	$-1.395(17)\cdot 10^{-11}$	$-1.396(17)\cdot 10^{-11}$	$-1.396(17)\cdot 10^{-11}$
14	$-5.361(67)\cdot 10^{-7}$	$-5.918(73)\cdot 10^{-7}$	$-5.756(72)\cdot 10^{-7}$
20	$-2.233(28)\cdot 10^{-6}$	$-2.713(33)\cdot 10^{-6}$	$-2.560(32)\cdot 10^{-6}$
54	$-1.187(15)\cdot 10^{-4}$	$-4.445(48)\cdot 10^{-4}$	$-2.706(34)\cdot 10^{-3}$
82	$-6.309(79)\cdot 10^{-4}$	$-1.413(11)\cdot 10^{-2}$	$-3.693(46)\cdot 10^{-3}$

Table 1. Results for the hadronic energy shift of the $1s$ ground state in hydrogen-like ions [2].

Conclusions

The rising precision of experimental spectroscopic measurements and theoretical predictions calls for more detailed description of known effects. This work is a contribution to understand and diminish the theoretical uncertainty induced by hadronic vacuum polarization in precision spectroscopy. The main conclusions are:

- For a broad range of hydrogen-like ions, hadronic effects are considerably larger than for the free electron.
- Hadronic effects will be observable in future bound-electron g factor experiments once nuclear charge radii and charge distributions will be substantially better known.
- Hadronic effects do not pose a limitation on testing QED or physics beyond the Standard Model, and determining fundamental constants through specific differences of g factors for different ions, or through the reduced g factor.

Hadronic Uehling potential

For hadronic states in the vacuum loop, the vacuum polarization function can be constructed semi-empirically from experimental data of e^-e^+ annihilation cross sections [2]. This effective hadronic polarization function accounts for all strong force interactions and is parametrized for seven regions of momentum transfer [3]. However, it was found that only the low-energy region of the parametrization is significant for atomic physics calculations. For a point-like nucleus with atomic number Z, the hadronic Uehling potential is given by [2]

$$\delta V^{\rm had.}_{\rm point}(r) = -\frac{2Z\alpha}{r}B_1E_1\left(\frac{r}{\sqrt{C_1}}\right) \, ,$$

where α is the fine-structure constant and $E_1(x)$ is the exponential integral. The coefficients are $B_1 = 0.002309(22)$ and $C_1 = 3.99(9)$ GeV^{-2} [2]. The hadronic Uehling potential for an extended nucleus can be found analytically by convolution with the nuclear charge distribution.

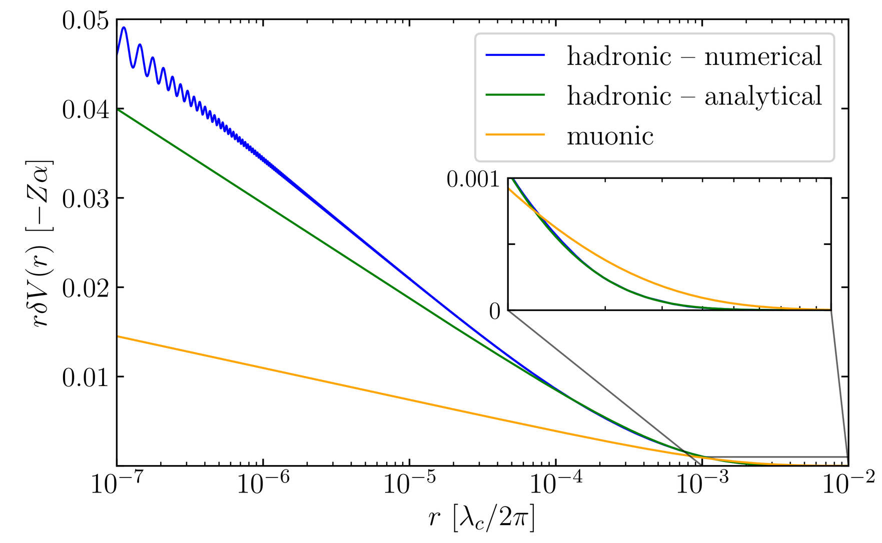

Figure 2. Comparison of muonic and hadronic Uehling potential for point-like nucleus [2].

Hadronic effects become more important close to the nucleus center. The oscillation in the numerical hadronic Uehling potential is due to finite integration boundaries.

Hadronic g factor correction

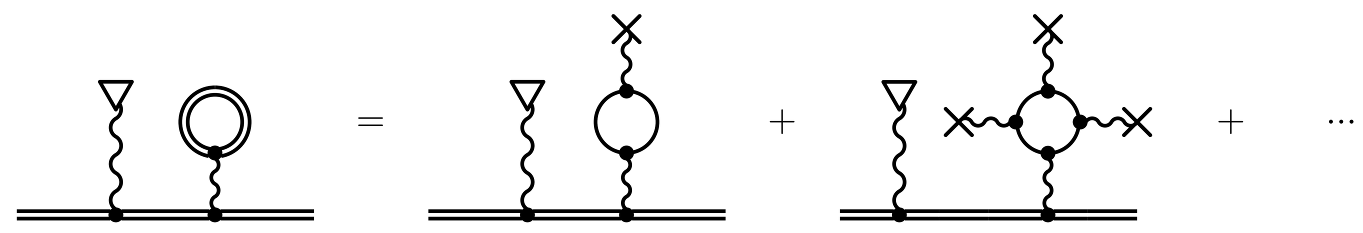

Figure 4. Vacuum polarization contribution to the bound-electron g factor [4].

Analytical results

The g factor describes the coupling of the electron's magnetic moment $\boldsymbol{\mu}$ to its total angular momentum $\boldsymbol{J}$. The first-order Zeeman splitting ΔE due to the electron's interaction with an external homogeneous magnetic field $\boldsymbol{B}$ is

$$\Delta E = -\langle \boldsymbol{\mu}\cdot\boldsymbol{B}\rangle = g\,\mu_{\rm B}\,\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle \, .$$

Having a small perturbation $\delta V(r)$ to the nucleus potential and using the Dirac equation, the g factor correction can be shown to be [4]

$$\Delta g = -\frac{\kappa^2}{j(j+1)m_e}\left\langle r\frac{\partial \delta V(r)}{\partial r}\right\rangle \, .$$

For light ions ($Z\alpha \ll 1$), the g factor shift of the $1s$ ground state can be approximated by

$$\Delta g(1s) \approx -\frac{4\,(1+2\gamma)}{3m_e}\Delta E(1s) \, .$$

Numerical results

The above formulas were used to calculate the hadronic g factor shifts for point-like and extended nuclei. Fig. 5 shows some results for the $1s$ ground state in hydrogen-like ions.

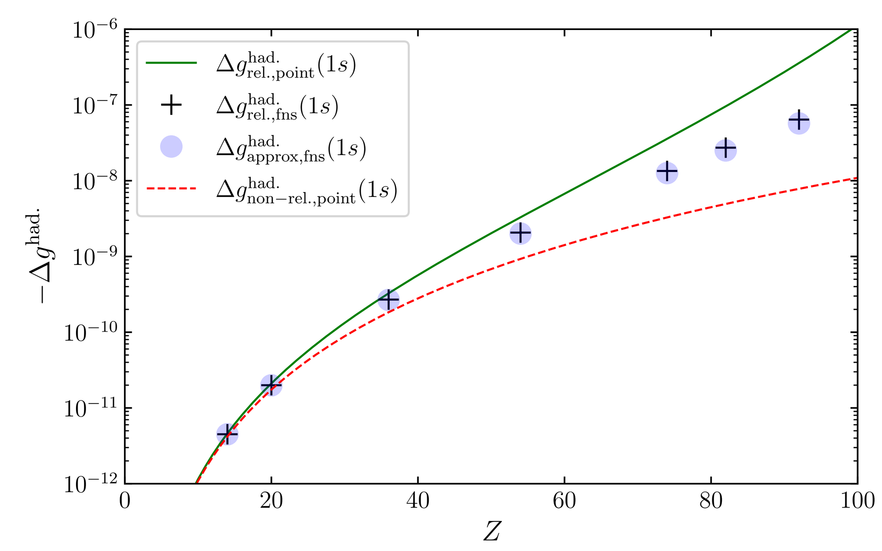

Figure 5. Comparison of analytical and numerical results for the hadronic g factor shift [5].

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