Gauge Fields, Knots and Gravity

Part I

Excersice 1

Let $\vec{\mathcal{E}}(t,\vec{x}) = \vec{E}\, e^{-i(\omega t - \vec{k}\cdot\vec{x})}$.

$$\vec{\nabla}\cdot\vec{\mathcal{E}} = i E_1 k_1 e^{-i(\omega t-\vec{k}\vec{x})} + i E_2 k_2 e^{-i(\omega t-\vec{k}\vec{x})} + i E_3 k_3 e^{-i(\omega t-\vec{k}\vec{x})} = i \underbrace{\vec{E}\cdot\vec{k}}_{=0} e^{-i(\omega t-\vec{k}\vec{x})} = 0$$

$$\vec{\nabla}\times\vec{\mathcal{E}} = \dots = \underbrace{i\vec{k}\times\vec{E}}_{=\omega\vec{E}} e^{-i(\omega t-\vec{k}\vec{x})} = i\cdot(-i\omega)\vec{E} e^{-i(\omega t-\vec{k}\vec{x})} = i\cdot\frac{\partial\vec{\mathcal{E}}}{\partial t}.$$

Excersice 2

„$\Rightarrow$" Assume f is continuous according to the topological definition.
Consider the ε-neighbourhood $B_\varepsilon(f(x))$ around $f(x)$. By the continuity of f, $f^{-1}(B_\varepsilon(f(x)))$ is open. This open set contains a neighbourhood about each of its points. Thus there exists a $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$.
That is $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. Since x was arbitrary choosen, f is continuous via the ε-δ-definition.

„$\Leftarrow$" Suppose that f is continuous via the ε-δ-definition.
Let U be open. By hypothesis f is continuous at every $x \in f^{-1}(U)$.
Thus there is a $\delta_x > 0$ such that $f(B_{\delta_x}(x)) \subseteq U$.
Thus $B_{\delta_x}(x) \subseteq f^{-1}(U)$, so that $f^{-1}(U) = \bigcup_{x\in f^{-1}(U)} B_{\delta_x}(x)$. Thus $f^{-1}(U)$ is open. □

Excersice 3

Define a collection of open sets U_α which covers S^n.
Define the chart maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ (projection) which are continuous and smooth.
Since the inverse maps are also continuous, the transition maps are smooth.

Excersice 4

If $A = \{(U_i, \varphi_i)\}_i$ is an atlas for M, then $A' = \{(U\cap U_i, \varphi_i')\}_i$ is an Atlas for U, where $\varphi_i' = \varphi_i|_{U\cap U_i}$.
Since $U \cap U_i$ is open, $\varphi_i' \circ (\varphi_j')^{-1}$ will be smooth as well for any charts $(U\cap U_i, \varphi_i')$ and $(U\cap U_j, \varphi_j')$ in A'. Hence U is also a manifold.

Excersise 5

If $A = \{(U_i, \varphi_i)\}_i$ is an atlas for X and $B = \{(V_j, \psi_j)\}_j$ is an atlas for Y, let $C = \{(U_i \times V_j, \varphi_i \times \psi_j)\}_{i \times j}$. We show that C is a valid atlas for $X \times Y$.

In the definition of C, $U_i \times V_j$ is the ordinary cartesian product. By $\varphi_i \times \psi_j$ we denote the map $\varphi_i \times \psi_j : X \times Y \to \mathbb{R}^{m+n}$ defined by $(\varphi_i \times \psi_j)(x, y) = (\varphi_i(x), \psi_j(y))$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$.

The charts certainly form an open cover for $X \times Y$. It remains to check that the transition maps are smooth. But $(\varphi_i \times \psi_j) \circ (\varphi_k \times \psi_\ell)^{-1}(x, y) = (\varphi_i \circ \varphi_k^{-1}(x), \psi_j \circ \psi_\ell(y))$ which is smooth since the component functions are smooth by definition.

Thus $X \times Y$ is a $(m+n)$-dimensional manifold.

Excersice 6

As before, let $A = \{(U_i, \varphi_i)\}_i$ be an atlas for X and $B = \{(V_j, \psi_j)\}_j$ be an atlas for Y.

Then $A \cup B$ is trivially an atlas for $X \cup Y$ since the collections of U_i's and V_j's over X and Y respectively, and since $U_i \cap V_j = \emptyset$ for any such charts, the transition functions only exist on X or Y seperately, hence being smooth by definition.

Thus $X \cup Y$ is a n-dimensional manifold.

Excersice 7

To show that $v+w$ and $gw \in \mathrm{Vect}(M)$, you have to show that they fullfill the 3 conditions

(i) $(v+w)(f+g) = v(f+g) + w(f+g) = v(f) + v(g) + w(f) + w(g) = (v+w)(f) + (v+w)(g)$

(ii) $(v+w)(\alpha f) = v(\alpha f) + w(\alpha f) = \alpha v(f) + \alpha w(f) = \alpha (v+w)(f)$

(iii) $(v+w)(f \cdot g) = v(fg) + w(fg) = v(f)g + f v(g) + w(f)g + f w(g) = (v+w)(f)g + (v+w)(g) \cdot f$

Analog for gw!

Excersice 8

$[f \cdot (v+w)](g) = f(v+w)(g) = f \cdot (v(g) + w(g)) = f \cdot v(g) + f w(g) = [fv + fw](g) \quad \forall g \in C^\infty(M)$

$[(f+g)v](h) = (f+g)v(h) = f v(h) + g v(h) = [fv + gv](h) \quad \forall h \in C^\infty(M)$

$[(fg)v](h) = (f \cdot g) v(h) = f \cdot (g v(h)) = [f(gv)](h) \quad \forall h \in C^\infty(M)$

$[1v](f) = 1 \cdot v(f) = v(f) \quad \forall f \in C^\infty(M)$

Excersice 9

Since $v^\mu \partial_\mu f = 0$ for all $f \in C^\infty(\mathbb{R}^n)$, choose $f = x^i$, $i \in \{1, \dots, n\}$.

Then $v^\mu \partial_\mu x^i = v^\mu \delta^i_\mu = v^i = 0$ for all $i \in \{1, \dots, n\}$.

Excersice 10

„$\Rightarrow$" Let $v = w$. Then $(v-w) = 0$ the null-vectorfield $\Rightarrow$ $(v-w)(f) = 0$ $\forall f \in C^\infty(M)$

$\Rightarrow$ $(v-w)(f)(p) = 0$ $\forall p \in M$

$\Rightarrow$ $v_p = w_p$ $\forall p \in M$

„$\Leftarrow$" Let $v_p = w_p$ $\forall p \in M$. Then $v(f)(p) - w(f)(p) = 0$ $\forall f \in C^\infty(M)$ and $\forall p \in M$

$\Rightarrow$ $v(f) - w(f) = 0$ $\forall C^\infty(M)$

$\Rightarrow$ $v = w$

Excersice 11

T_pM is a vector space over the real numbers:

Let $u, v, w \in T_pM$ and $\alpha, \beta \in \mathbb{R}$.

- $u + (v + w) = (u + v) + w$

  $v + w = w + v$

follow from the definition of addition and $\mathbb{R}$ beeing a commutative group

- zero vector: $0 \in T_pM$ by $0(f) = 0$ for all $f \in C^\infty(M)$
- additive inversion: $(-v)(f) = -v(f)$
- distributive laws:

$$(\alpha(v+w))(f) = \alpha((v+w)(f)) = \alpha(v(f) + w(f)) = \alpha v(f) + \alpha w(f)$$
$$= (\alpha v)(f) + (\alpha w)(f) = (\alpha v + \alpha w)(f)$$

$$((\alpha+\beta)v)(f) = (\alpha+\beta)v(f) = \alpha v(f) + \beta v(f) = (\alpha v)(f) + (\beta v)(f)$$
$$= (\alpha v + \beta v)(f)$$

$(\alpha\beta)v = \alpha(\beta v)$ follows from the properties of $\mathbb{R}$

- 1-Element: $1 \in T_pM$ by $1(f) = 1$ for all $f \in C^\infty(M)$

Then $(1v)(f) = 1 \cdot v(f) = v(f)$!

Excersice 12

$\gamma'(t) : C^\infty(M) \longrightarrow \mathbb{R} \quad , \quad f \longmapsto \frac{d}{dt} f(\gamma(t))$

(i) $\gamma'(t)(f+g) = \frac{d}{dt}(f+g)(\gamma(t)) = \frac{d}{dt} f(\gamma(t)) + \frac{d}{dt} g(\gamma(t)) = \gamma'(t)(f) + \gamma'(t)(g)$

(ii) $\gamma'(t)(\alpha f) = \frac{d}{dt}(\alpha f)(\gamma(t)) = \alpha \frac{d}{dt} f(\gamma(t)) = \alpha \gamma'(t)(f)$

(iii) $\gamma'(t)(f \cdot g) = \frac{d}{dt}(f \cdot g)(\gamma(t)) = \frac{d}{dt}\left[f(\gamma(t)) \cdot g(\gamma(t))\right] = \frac{d}{dt} f(\gamma(t)) \cdot g(\gamma(t)) + f(\gamma(t)) \cdot \frac{d}{dt} g(\gamma(t))$

$= \gamma'(t)(f) \cdot g(\gamma(t)) + f(\gamma(t)) \cdot \gamma'(t)(g)$

$\Rightarrow \gamma'(t) \in T_{\gamma(t)} M$.

Excersice 13

$\phi : \mathbb{R} \to \mathbb{R} \quad , \quad t \mapsto \phi(t) = e^t$.

$\phi^* x = x \circ \phi = e^t$

Excersice 14

$\phi : \mathbb{R}^2 \to \mathbb{R}^2 \quad , \quad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\phi^* x = x \circ \phi = \cos\theta \, x - \sin\theta \, y$

$\phi^* y = y \circ \phi = \sin\theta \, x + \cos\theta \, y$

Excersice 15

Consider smooth functions $f : M \to \mathbb{R}$.

„$\Rightarrow$" Let $f : M \to \mathbb{R}$ be any function such that $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth for all α .
Because the composition of smooth functions is smooth, we have $g \circ (f \circ \phi_\alpha^{-1}) = (g \circ f) \circ \phi_\alpha^{-1}$
is smooth for all α and any $g \in C^\infty(M)$. This implies that $g \circ f$ is smooth.

„$\Leftarrow$" Assume that f is smooth according to the new definition.
$\Rightarrow$ For any $g \in C^\infty(M)$, we have $g \circ f \in C^\infty(M)$.
Take $g = id_{\mathbb{R}}$, then it follows that $f \in C^\infty(M)$ is smooth, which is the old definition.

Same arguments for smooth curves.

Excersice 16

$$(\phi\circ\gamma)'(t) = \frac{d}{dt}(f((\phi\circ\gamma)(t)) = \frac{d}{dt}(f\circ\phi\circ\gamma)(t) = \frac{d}{dt}(f\circ\phi)(\gamma(t)) = \gamma'(t)(\phi^*f) = \phi_*(\gamma'(t))$$

Excersice 17

Let $v, w \in T_pM$ and $f \in C^\infty(M)$. Then

$$\phi_*(v+w)(f) = (v+w)(\phi^*f) = v(\phi^*f) + w(\phi^*f) = (\phi_*v)(f) + (\phi_*w)(f)$$
$$= (\phi_*v + \phi_*w)(f)$$

Let $\alpha \in \mathbb{R}$. Then

$$\phi_*(\alpha v)(f) = (\alpha v)(\phi^*f) = \alpha\, v(\phi^*f) = \alpha\, \phi_*v(f) = (\alpha(\phi_*v))(f)$$

Excersice 18

This is fullfilled, if $\phi_*v : C^\infty(M) \to C^\infty(M)$, $f \mapsto v(f\circ\phi)\circ\phi^{-1}$.

It is easy to check that this is a vectorfield on N. Then

$$(\phi_*v)_q(f) = [\phi_*v(f)](q) = [v(f\circ\phi)\circ\phi^{-1}](q) = [v(f\circ\phi)](p) = [v(\phi^*f)](p)$$
$$= (\phi_*(v_p))(f)$$

Excersice 19

Let $\phi(x,y) = (u(x,y), v(x,y))$ with $u(x,y) = \cos\theta\, x - \sin\theta\, y$ and $v(x,y) = \sin\theta\, x + \cos\theta\, y$.

$$(\phi_*\partial_x)(f) = \partial_x(\phi^*f) = \frac{\partial}{\partial x}(f\circ\phi) = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = (\partial_x f)\cos\theta + (\partial_y f)\sin\theta$$
$$= (\cos\theta\,\partial_x + \sin\theta\,\partial_y)(f)$$

$$(\phi_*\partial_y)(f) = \partial_y(\phi^*f) = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = (\partial_x f)(-\sin\theta) + (\partial_y f)\cos\theta$$
$$= (-\sin\theta\,\partial_x + \cos\theta\,\partial_y)(f)$$

Excersice 20

Setting $\gamma(t) = (x(t), y(t))$, we have with $\gamma'(t) = v_{\gamma(t)}$ and $v = x^2\partial_x + y\,\partial_y$;

$$x'\partial_x + y'\partial_y = x^2\partial_x + y\,\partial_y \iff \begin{array}{ll}(i) & x' = x^2\\ (ii) & y' = y\end{array}$$

The solution depends on $p = (a,b)$,

1. $a=0, b=0 \Rightarrow x(t)=0, \; y(t)=0$
2. $a\neq0, b=0 \Rightarrow x(t)=\frac{a}{1-at}, \; y(t)=0$
3. $a=0, b\neq0 \Rightarrow x(t)=0, \; y(t)=b\cdot e^t$
4. $a\neq0, b\neq0 \Rightarrow x(t)=\frac{a}{1-at}, \; y(t)=b\,e^t$.

Excersice 21

$\phi_0 : X \to X$, $p \mapsto p$ identity map follows from the definition.

For the second part define $\psi(t) : \mathbb{R} \to \mathbb{R}$, $t \mapsto s+t$. Let $q = \phi_s(p)$.

Consider two curves:
- $\phi_t(q)$ with properties $\phi_0(q) = q$ and $\frac{d}{dt} \phi_t(q) \Big|_{t=0} = v_q$
- $\phi_s(p) \circ \psi(t)$

Note that $(\phi_s(p) \circ \psi)(t) = \phi_{\psi(t)}(p)$, so $(\phi_s(p) \circ \psi)(0) = \phi_s(p) = q$

and $\frac{d}{dt}(\phi_s(p) \circ \psi)(t) \Big|_{t=0} = \frac{d}{ds} \phi_s(p) \Big|_{s=s} \cdot \frac{d}{dt} \psi(t) \Big|_{t=0} = v_{\phi_s(p)} \cdot 1 = v_q$.

Thus both functions have the same value and derivative at $t=0$.

$$\Rightarrow \quad \phi_t(q) = (\phi_s(p) \circ \psi)(t)$$

$$\phi_t(\phi_s(p)) = \phi_{\psi(t)}(p)$$

$$(\phi_t \circ \phi_s)(p) = \phi_{s+t}(p).$$

Excersice 22

With $v = \frac{x \partial_x + y \partial_y}{\sqrt{x^2+y^2}}$ and $w = \frac{x \partial_y - y \partial_x}{\sqrt{x^2+y^2}}$ we get $[v,w] = \frac{y \partial_x - x \partial_y}{x^2+y^2}$

Excersice 23

$$(vf)(p) = \frac{d}{dt} f(\phi_t(p)) \Big|_{t=0}$$

$$(wf)(p) = \frac{d}{ds} f(\psi_s(p)) \Big|_{s=0}$$

$$\Rightarrow$$

$$\begin{aligned} [v,w](f)(p) &= v(w(f))(p) - w(v(f))(p) \\ &= \frac{d}{dt}(w(f))(\phi_t(p)) \Big|_{t=0} - \cdots\cdots \\ &= \frac{d}{dt}\left(\frac{\partial}{\partial s} f(\psi_s(\phi_t(p))) \Big|_{s=0} \right) \Big|_{t=0} - \cdots \\ &= \frac{\partial^2}{\partial t \partial s} f(\psi_s(\phi_t(p))) \Big|_{s=t=0} - \cdots \end{aligned}$$

Excersice 24

1) $[v,w] = vw - wv = -(wv - vw) = -[w,v]$

2) $[u, \alpha v + \beta w] = u(\alpha v + \beta w) - (\alpha v + \beta w) u = \alpha uv + \beta uw - \alpha vu - \beta wu$

$$= \alpha(uv - vu) + \beta(uw - wu)$$

$$= \alpha[u,v] + \beta[u,w]$$

3) $[u,[v,w]] + [v,[w,u]] + [w,[u,v]]$

$= u[v,w] - [v,w]u + v[w,u] - [w,u]v + w[u,v] - [u,v]w$

$= uvw - uwv - vwu + wvu + vwu - vuw - wuv + uwv + wuv - wvu - uvw + vuw$

$= 0$

Excersice 25

$$(\omega+\mu)(v+w) = \omega(v+w) + \mu(v+w) = \omega(v) + \omega(w) + \mu(v) + \mu(w) = (\omega+\mu)(v) + (\omega+\mu)(w)$$

$$(\omega+\mu)(gv) = \omega(gv) + \mu(gv) = g\,\omega(v) + g\mu(v) = g\cdot(\omega+\mu)(v)$$

Analog for $f\omega$!

Excersice 26

For all $\omega, \mu \in \Omega^1(M)$ and $f, g \in C^\infty(M)$ we have :

$$[f(\omega+\mu)](v) = f(\omega+\mu)(v) = f\cdot(\omega(v)+\mu(v)) = [f\omega](v) + [f\mu](v) = [f\omega + f\mu](v)$$

$$[(f+g)\omega](v) = (f+g)\cdot\omega(v) = f\omega(v) + g\omega(v) = [f\omega + g\omega](v)$$

$$[(fg)\omega](v) = (fg)\omega(v) = f\cdot(g\omega(v)) = f\cdot(g\omega)(v) = [f(g\omega)](v)$$

$$[1\omega](v) = 1\cdot\omega(v) = \omega(v)$$

Excersice 27

$$d(f+g)(v) = v(f+g) = v(f) + v(g) = df(v) + dg(v) = [df + dg](v)$$

$$d(\alpha f)(v) = v(\alpha f) = \alpha\, v(f) = \alpha\, df(v) = [\alpha\, df](v)$$

$$[(f+g)\,dh](v) = (f+g)\,dh(v) = f\,dh(v) + g\,dh(v) = [f\,dh + g\,dh](v)$$

$$d(fg)(v) = v(f\cdot g) = v(f)\,g + f\,v(g) = df(v)\,g + f\,dg(v) = [g\,df + f\,dg](v)$$

Excersice 28

To show that $df = \partial_\mu f\, dx^\mu$, we use the fact that any vector field v on $\mathbb{R}^n$ is of the form $v = f'(x^1, \dots, x^n)\,\partial_{\mu'}$. Then :

- $df(v) = v(f) = f'\,\partial_{\mu'} f$
- $(\partial_\mu f\, dx^\mu)(v) = (\partial_\mu f)\, dx^\mu(v) = \partial_\mu f \cdot f'\,\partial_{\mu'} x^\mu = \partial_\mu f \cdot f'\,\delta^\mu_{\mu'} = f'\cdot\partial_{\mu'} f$

$\Rightarrow \quad df(v) = (\partial_\mu f\, dx^\mu)(v) \quad \forall v \in \mathrm{Vect}(M)$

$\Rightarrow \quad df = \partial_\mu f\, dx^\mu$

Excersice 29

If $\omega = \omega_\mu dx^\mu = 0$, then it's equal zero for all $v \in \mathrm{Vect}(M)$. Choose $v = \partial_\nu$:

$$\Rightarrow \quad \omega(v) = \omega_\mu\, dx^\mu(\partial_\nu) = \omega_\mu\, \partial_\nu x^\mu = \omega_\mu\, \delta^\mu_\nu = \omega_\nu = 0 \qquad \forall \nu .$$

Excersice 30 :

We show : If v and w are two vector fields such that $v_p = w_p \Rightarrow \omega(v)(p) = \omega(w)(p)$.

Let $z = v - w$, then $z_p = v_p - w_p = 0$ and $\omega(z)(p) = 0$ would imply $\omega(v)(p) = \omega(w)(p)$.

$\Rightarrow$ it is enough to prove that a vector field v with $v_p = 0$, it follows that $\omega(v)(p) = 0$.

Choose local coordinates so $v_p = (v^\nu \partial_\nu)_p$.

$$\Rightarrow \quad v_p(x^\mu) = (vx^\mu)(p) = (v^\nu \partial_\nu x^\mu)(p) = v^\mu(p)$$

But since $v_p = 0$, the functions v^μ all vanish at p.

Also we have $\omega(v)(p) = (\omega_\mu dx^\mu)(v^\nu \partial_\nu)(p) = (\omega_\mu v^\mu)(p) = \omega_\mu(p) v^\mu(p) = 0$ ✓

It is unique : $\omega(v)(p) = \omega_p(v_p) = \nu_p(v_p) = \nu(v)(p)$. $\square$

Excersice 31 :

Let $id_V : V \to V$ be the identity map on V. We have to show $id_V^* : V^* \to V^*$ is identity on V^*.

Let $f \in V^*$, so $(id_V^* f)(v) = f(id_V(v)) = f(v) \Rightarrow id_V^*(f) = f$ ✓

Moreover : $(f^* g^* x)(v) = (g^* x)(f(v)) = x(g(f(v))) = x(gf(v)) = ((gf)^* x)(v)$ $\square$

Excersice 32 :

Existence : Is the function $p \mapsto \phi^*\omega(v)(p) = \omega(\phi_*(v_p))$ a smooth function on M ?

If x^μ is a chart around p and y^α is a chart around $\phi(p)$, then :

$$x \mapsto \omega(\phi_*(v(x))) = \omega_\alpha dy^\alpha(\phi_*(v^\mu(x)\partial_\mu)) = \omega_\alpha dy^\alpha\left(v^\mu(x) \frac{\partial y^\beta}{\partial x^\mu} \partial_\beta\right) = \omega_\beta(\phi(x)) v^\mu(x) \frac{\partial y^\beta}{\partial x^\mu}$$

which is a smooth function since ω_β, v^μ and the Jacobian $\frac{\partial y^\beta}{\partial x^\mu}$ are smooth.

Uniqueness : by definition (from ex 30). $\square$

Excersice 33 :

$$\phi^* dx = d(\phi^* x) = d(\sin t) = \left(\frac{\partial}{\partial t} \sin t\right) dt = \cos t \, dt$$

Excersice 34 :

We have $\phi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta\, x - \sin\theta\, y \\ \sin\theta\, x + \cos\theta\, y \end{pmatrix}$ rotation counterclockwise by the angle θ.

$$\Rightarrow \quad \phi^* dx = d(\phi^* x) = d(\cos\theta\, x - \sin\theta\, y) = \cos\theta\, dx - \sin\theta\, dy$$

$$\phi^* dy = d(\phi^* y) = d(\sin\theta\, x + \cos\theta\, y) = \sin\theta\, dx + \cos\theta\, dy$$

Excersice 35 :

$$dx^\mu(\partial_\nu) = \partial_\nu(x^\mu) = \delta^\mu_\nu$$

$$(\phi^* dx^\mu)((\phi^{-1})_*\partial_\nu) = dx^\mu(\phi_*(\phi^{-1})_*\partial_\nu) = dx^\mu(\partial_\nu) = \delta^\mu_\nu$$

Excersice 36 :

$$dx'^\nu = T^\nu_\mu \, dx^\mu$$

$$\Rightarrow \quad dx'^\nu(\partial_\lambda) = T^\nu_\mu \, dx^\mu(\partial_\lambda)$$

$$\Leftrightarrow \quad \frac{\partial x'^\nu}{\partial x^\lambda} = T^\nu_\lambda$$

$$\Rightarrow \quad dx'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} dx^\mu$$

$$\omega = \omega'_\mu dx'^\mu = \omega_\mu dx^\mu$$

$$\Rightarrow \quad \omega'_\mu \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu = \omega_\nu dx^\nu$$

$$\Rightarrow \quad \omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \omega_\nu$$

Because

$$\frac{\partial x'^\mu}{\partial x'^\mu} = \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{=\left[\frac{\partial x'^\mu}{\partial x^\nu}\right]^{-1}} \frac{\partial x'^\mu}{\partial x^\nu} = 1$$

Excersice 37 :

$$\left.\begin{aligned} \phi^*(dx'^\nu)(\partial_\lambda) &= dx'^\nu(\phi_*\partial_\lambda) = \frac{\partial x'^\nu}{\partial x^\lambda} \\ \frac{\partial x'^\nu}{\partial x^\mu} dx^\mu(\partial_\lambda) &= \frac{\partial x'^\nu}{\partial x^\mu}\delta^\mu_\lambda = \frac{\partial x'^\nu}{\partial x^\lambda} \end{aligned}\right\} \Rightarrow \quad \phi^*(dx'^\nu) = \frac{\partial x'^\nu}{\partial x^\mu} dx^\mu$$

Excersice 38 :

T must transform the basis $\partial_\mu|_p$ into some other basis $\rightarrow$ $T^\nu_\mu(p)$ must be invertible

Excersice 39 :

Uniqueness : Suppose g^μ are another set of 1-Forms satisfying $g^\mu(e_\nu) = \delta^\mu_\nu$.
Then clearey $f^\mu = g^\mu$, since a functional is determined by its action on a basis.

Existence : $f^\mu = (T^{-1})^\mu_\nu dx^\nu$

Since $(T^{-1})^\mu_\nu dx^\nu(e_\lambda) = (T^{-1})^\mu_\nu dx^\nu(T^\alpha_\lambda \partial_\alpha) = (T^{-1})^\mu_\nu T^\alpha_\lambda \delta^\nu_\alpha = (T^{-1})^\mu_\nu T^\nu_\lambda = \delta^\mu_\lambda$

Excersice 40 :

$f'^\mu = (T^{-1})^\mu_\nu f^\nu$ since both sides yield $f'^\mu(e'_\nu) = \delta^\mu_\nu$.

If $v = v^\mu e_\mu = v'^\mu e'_\mu$ then with $e'_\mu = T^\nu_\mu e_\nu$ it follows $v^\nu e_\nu = v'^\mu T^\nu_\mu e_\nu$

$$\Rightarrow \quad v'^\mu = (T^{-1})^\mu_\nu v^\nu.$$

If $w = w_\mu f^\mu = w'_\mu f'^\mu$ then $w_\nu f^\nu = w'_\mu (T^{-1})^\mu_\nu f^\nu \Rightarrow w'_\mu = T^\nu_\mu w_\nu$.

Excersice 41 :

Let

$$v = v_x dx + v_y dy + v_z dz$$
$$w = w_x dx + w_y dy + w_z dz$$
$$u = u_x dx + u_y dy + u_z dz$$

then

$$u \wedge v \wedge w = u_x(v_y w_z - w_y v_z)\, dx \wedge dy \wedge dz + u_y(w_x v_z - v_x w_z)\, dx \wedge dy \wedge dz + u_z(v_x w_y - w_x v_y)\, dx \wedge dy \wedge dz$$

which is the same as

$$u \wedge v \wedge w = \underbrace{\det\begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}}_{= \vec{u}\cdot(\vec{v}\times\vec{w})} dx \wedge dy \wedge dz$$

Excersice 42 :

If a, b, c, d are four vectors in a 3-dimensional space, then

$$a \wedge b \wedge c \wedge d = \det \begin{pmatrix} a_x & a_y & a_z & 0 \\ b_x & b_y & b_z & 0 \\ c_x & c_y & c_z & 0 \\ d_x & d_y & d_z & 0 \end{pmatrix} dx \wedge dy \wedge dz = 0$$

Or : Because there are four vectors and only three basis vectors, in every term will appear a duble basis vector. Because of antisymmetry this vanishes.

Excersice 43 :

$\wedge V$ if :
V is 1-dimensional : $v = v_x dx \in V$ $\wedge V = \{ v_1 \}$
V is 2-dimensional : $v = v_x dx + v_y dy \in V$ $\wedge V = \{ v_1 \wedge v_2 \}$
V is 4-dimensional : $V = v_x dx + v_y dy + v_z dz + v_w dw \in V$ $\wedge V = \{ v_1 \wedge v_2 \wedge v_3 \wedge v_4 \}$

Excersice 44 :

If $p > n$ then $\wedge^p V$ is empty or just zero because at least one basis vector appears twice. For $0 \leq p \leq n$ the dimension of $\wedge^p V$ is $n!/p!(n-p)!$, since you have to choose p vectors out of n possible basis vectors $\Rightarrow \binom{n}{p} = \frac{n!}{p!(n-p)!}$ options, these buied the basis for $\wedge^p V$.

Excersice 45 :

$\wedge V$ is the direct sum of the subspaces $\wedge^p V$ $(\wedge V = \oplus \wedge^p V)$ because every vector $v \in \wedge V$ can be expressed uniquely as $v_1 + \dots + v_n$ where $v_i \in \wedge^i V$.

The dimension of $\wedge V$ is therefore the sum of the dimensions of all subrooms $\wedge^p V$:

$$\begin{aligned} \dim \wedge V = \sum_{i=0}^{n} \binom{n}{i} &= 1 + n + \frac{n \cdot (n-1)}{2} + \dots + \frac{n \cdot (n-1)}{2} + n + 1 \\ &= 2 + 2n + n(n-1) + \dots \\ &= 2^n \end{aligned}$$

Induction : $\sum_{i=0}^{n+1} \binom{n+1}{i} = \sum_{i=0}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1} \right] = 2^n + \sum_{i=0}^{n+1} \binom{n}{i-1} = 2 \cdot 2^n = 2^{n+1}$

Excersice 46 :

$$\left. \begin{aligned} \text{If } \omega \in \wedge^p V &\Rightarrow \omega_1 \wedge \dots \wedge \omega_p = \omega \\ \mu \in \wedge^q V &\Rightarrow \mu_1 \wedge \dots \wedge \mu_q = \mu \end{aligned} \right\} \quad \begin{aligned} \omega \wedge \mu &= \omega_1 \wedge \dots \wedge \omega_p \wedge \mu_1 \wedge \dots \wedge \mu_q \\ &= (-1)^p \mu_1 \wedge \omega_1 \wedge \dots \wedge \omega_p \wedge \mu_2 \wedge \dots \wedge \mu_q \\ &= (-1)^{pq} \mu_1 \wedge \dots \wedge \mu_q \wedge \omega_1 \wedge \dots \wedge \omega_p \\ &= (-1)^{pq} \mu \wedge \omega \end{aligned}$$

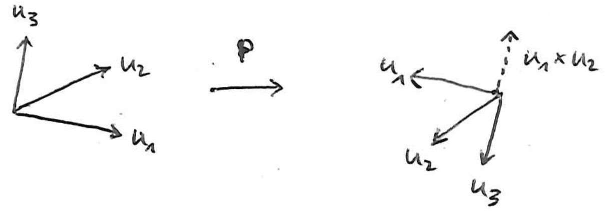
Excersice 47 :

usual Pullback map as for functions.

Excersice 48 :

If ω is a 1-form then $P(\omega) = -\omega$

If μ is a 2-form then $P(\mu) = \mu$



$$\phi^*(\omega_\mu dx^\mu) = (\omega_\mu \circ \phi)\, dx^\mu$$

$$\phi^*\left(\tfrac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu\right) = \tfrac{1}{2}(\omega_{\mu\nu} \circ \phi)\, dx^\mu \wedge dx^\nu$$

Excersice 49 :

$$d(\omega_\mu dx^\mu) = d\omega_\mu \wedge dx^\mu = \partial_\nu \omega_\mu dx^\nu \wedge dx^\mu$$

We used here the Leibniz rule $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^p \omega \wedge d\mu \quad \forall \omega \in \Omega^p(M), \mu \in \Omega(M)$.

Since ω_μ is a function $\Rightarrow \omega_\mu \in \Omega^0(M)$

$$\Rightarrow d(\omega_\mu dx^\mu) = d(\omega_\mu \wedge dx^\mu) = d\omega_\mu \wedge dx^\mu + (-1)^0 \omega_\mu \wedge \underbrace{d(dx^\mu)}_{=0} = d\omega_\mu \wedge dx^\mu.$$

Excersice 50 :

We can write any 2-form as $F = B + E \wedge dt$ because $\{(dx^i)_p \wedge (dx^j)_p, (dx^k)_p \wedge dt\}_{i,j,k=1,\dots,\dim S}$ span $\Lambda^2 T^*_{(t,p)}(\mathbb{R} \times S)$.

Uniqueness : Suppose that $F = B' + E' \wedge dt$. Locally we can write

$$F = \tfrac{1}{2} B_{ij}\, dx^i \wedge dx^j + E_i\, dx^i \wedge dt = \tfrac{1}{2} B'_{ij}\, dx^i \wedge dx^j + E'_i\, dx^i \wedge dt$$

Now the forms $\{dx^i \wedge dx^j, dx^i \wedge dt\}$ are lineary independent, so we must have $B_{ij} = B'_{ij}$ and $E_i = E'_i$.

Excersice 51 :

$$d\omega = d(\omega_I dx^I) = d\omega_I \wedge dx^I = \partial_0 \omega_I dx^0 \wedge dx^I + \partial_i \omega_I dx^i \wedge dx^I = dt \wedge \partial_t \omega + d_S \omega$$

Excersice 52 :

The map from V to V^* given by $v \mapsto g(v,\cdot)$ is an isomorphism.

One-to-one : Suppose $g(v,\cdot) = g(w,\cdot) \Rightarrow g(v-w,\cdot) = 0$

Since this function is zero for every $x \in V \Rightarrow v - w = 0 \Rightarrow v = w$.

Onto : Since the dimensions of V and V^* are equal.

Excersice 53 :

Let $v = v^\mu e_\mu$. The corresponding 1-Form $g(v,\cdot)$ can be expressed in the dual basis $g(v,\cdot) \overset{!}{=} a_\nu f^\nu$. We can find the coefficients by its action on a basis element

$$\left.\begin{aligned} g(v, e_\nu) &= g(v^\mu e_\mu, e_\nu) = v^\mu g_{\mu\nu} \\ a_\mu f^\mu(e_\nu) &= a_\mu \delta^\mu_\nu = a_\nu \end{aligned}\right\} \quad a_\nu = v^\mu g_{\mu\nu} \qquad \Rightarrow v_\mu = g_{\mu\nu} v^\nu$$

And since

$$\left.\begin{aligned} g(v,v) &= v^\mu v^\nu g_{\mu\nu} = v^\mu v_\mu \\ a_\mu f^\mu(v) &= a_\mu v^\nu \delta^\mu_\nu = a_\mu v^\mu \end{aligned}\right\} \quad a_\mu = v_\mu \qquad \text{„ rename } a_\mu \to v_\mu \text{"}$$

Excersice 54 :

Let $\omega = \omega_\mu f^\mu$. The corresponding vector field is $\omega^\nu e_\nu$.

We know that $\omega_\mu = g_{\mu\nu}\omega^\nu$ and since $g_{\mu\nu}$ is invertible : $\omega^\nu = g^{\mu\nu}\omega_\mu$.

Excersice 55 :

Let $\eta(v,w) = -v^0w^0 + v^1w^1 + v^2w^2 + v^3w^3$ be the standard Minkowskis metric.

Then we have for the standard basis e_μ : $\eta(e_0,e_0) = -1$ and $\eta(e_i,e_j) = \delta_{ij}$.

$$\Rightarrow \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Excersice 56 :

$g^\mu_\nu = g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu_\nu$ since $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$.

Excersice 57 :

Nondegenerate : If $\langle e^1 \wedge \dots \wedge e^p , f^1 \wedge \dots \wedge f^p \rangle = 0 \quad \forall f^1 \wedge \dots \wedge f^p \in \Lambda^p V$

Then $\det(\langle e^i, f^j \rangle) = 0 \quad \forall f^j \in \Lambda^1 V$

But since $\langle \omega, \mu \rangle$ is nondegenerate for 1-forms $\Rightarrow e^i = 0$

$\Rightarrow e^1 \wedge \dots \wedge e^p = 0$

$$\langle e^{i_1} \wedge \dots \wedge e^{i_p} , e^{i_1} \wedge \dots \wedge e^{i_p} \rangle = \det(\langle e^i, e^j \rangle) \underset{\substack{\uparrow \\ e^i \text{ ONB}}}{=} \det \begin{pmatrix} \varepsilon(i_1) & & 0 \\ & \ddots & \\ 0 & & \varepsilon(i_p) \end{pmatrix} = \varepsilon(i_1) \cdot \dots \cdot \varepsilon(i_p)$$

Excersice 58 :

Let $E = E_x dx + E_y dy + E_z dz$ be a 1-form on $\mathbb{R}^3$ with its Euclidean metric.

$$\langle E, E \rangle = g^{ij} E_i E_j = \delta^{ij} E_i E_j = E_x^2 + E_y^2 + E_z^2$$

And for $B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$ we get

$$\begin{aligned} \langle B, B \rangle &= B_x \langle dy \wedge dz , B \rangle + B_y \langle dz \wedge dx , B \rangle + B_z \langle dx \wedge dy , B \rangle \\ &= B_x^2 \langle dy \wedge dz , dy \wedge dz \rangle + B_x B_y \underline{\langle dy \wedge dz , dz \wedge dx \rangle} + B_x B_z \underline{\langle dy \wedge dz , dx \wedge dy \rangle} \\ &+ B_y^2 \langle dz \wedge dx , dz \wedge dx \rangle + B_y B_x \underline{\langle dz \wedge dx , dy \wedge dz \rangle} + B_y B_z \underline{\langle dz \wedge dx , dx \wedge dy \rangle} \\ &+ B_z^2 \langle dx \wedge dy , dx \wedge dy \rangle + B_z B_x \underline{\langle dx \wedge dy , dy \wedge dz \rangle} + B_z B_y \underline{\langle dx \wedge dy , dz \wedge dx \rangle} \\ &= B_x^2 + B_y^2 + B_z^2 \end{aligned}$$

Excersice 59 :

$$\begin{aligned} -\tfrac{1}{2}\langle F, F \rangle &= -\tfrac{1}{2}\left(\langle B, F \rangle + \langle E \wedge dt , F \rangle \right) = -\tfrac{1}{2}\left(\langle B, B \rangle + 2\langle E \wedge dt , B \rangle + \langle E \wedge dt , E \wedge dt \rangle \right) \\ &= -\tfrac{1}{2}\left(\langle B, B \rangle + (-1)\langle E, E \rangle \right) \\ &= \tfrac{1}{2}\left(\langle E, E \rangle - \langle B, B \rangle \right) \end{aligned}$$

$\Rightarrow$ Lagrangian !

Excersice 60 :

Let T be the transformation from basis $\{e_\mu\}$ to basis $\{e'_\mu\}$ that interchanges to basis elements, e.g. a transposition. A transposition matrix differs from the identity in the fact that two columns (or rows) are swapped. This leads to a minus sign in the determinant $\Rightarrow$ $\det T = -1$. If $P = T_1 \cdot \ldots \cdot T_n \Rightarrow \det P = (-1)^n$.

If we have odd permutations, so odd number of transpositions, we have opposite orientation. If we have even permutations, so even number of transpositions, we have the same orientation.

Excersice 61 :

In local coordinates $\omega = f\, dx^1 \wedge \ldots \wedge dx^n$, since ω is a volume form: $f \neq 0$ on $\varphi_\alpha(U_\alpha)$. So eather $f < 0$ or $f > 0$. In both cases, we can find charts so that ω is oriented pos.

Excersice 62 :

If we can cover M with charts such that the transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are orientation-preserving, we can make M into an oriented manifold by using the charts to transfer the standard orientation on $\mathbb{R}^n$ to an orientation on M.

The standard volume form on $\mathbb{R}^n$ is $dx^1 \wedge \ldots \wedge dx^n$. Locally we can define the volume form $\omega = \phi^{-1}(dx^1 \wedge \ldots \wedge dx^n)$ on M. Since ϕ is orientation preserving, ω is positively orientated. Since our transition functions are also orientation preserving, we can change the charts without changing the orientation of ω $\Rightarrow$ ω is globally positively orientated.

Excersice 63 :

The volume form associated to the metric on M in point p is $vol_p = \sqrt{|\det(g_{\mu\nu})_p|}\,(dx^1)_p \wedge \ldots \wedge (dx^n)_p$.

Let $\{e^\mu\}$ be an orientated ONB at the point p such that $(dx^\mu)_p = T^\mu_\nu e^\nu$.

Notice that $\langle (dx^\mu)_p, (dx^\nu)_p \rangle = (g^{\mu\nu})_p$ and $\langle (dx^\nu)_p, (dx^\mu)_p \rangle = T^\nu_\alpha T^\mu_\beta \langle e^\alpha, e^\beta \rangle = \pm T^2$

Taking the determinant on both equation yields:

$$\left.\begin{array}{ll} 1) & \det(g^{\mu\nu})_p = \det(g_{\mu\nu})_p^{-1} = \pm |\det(g_{\mu\nu})_p|^{-1} \\ 2) & \det\langle (dx^\mu)_p, (dx^\nu)_p \rangle = \pm |\det T^\mu_\lambda|^2 \end{array}\right\} \det T^\mu_\lambda = \sqrt{|\det(g_{\mu\nu})_p|}^{\,-1}$$

$$\begin{aligned} \Rightarrow \quad vol_p &= \sqrt{|\det(g_{\mu\nu})_p|}\,(dx^1)_p \wedge \ldots \wedge (dx^n)_p \\ &= \sqrt{|\det(g_{\mu\nu})_p|}\, T^1_\lambda e^\lambda \wedge \ldots \wedge T^n_\lambda e^\lambda \\ &= \sqrt{|\det(g_{\mu\nu})_p|}\, \det T^\mu_\lambda\, e^1 \wedge \ldots \wedge e^n \\ &= e^1 \wedge \ldots \wedge e^n . \end{aligned}$$

Excersice 64:

We choose a positively oriented ONB $\{e^\mu\}$ on some chart. Then the p-Forms can be expressed as $\omega = \omega_I e^I$ and $\mu = \mu_{I'} e^{I'}$, where I and I' are multi-indices. Since only equal basis terms will survive in $\langle \omega, \mu \rangle$, we look at the case where $\omega = f\, e^{i_1} \wedge \dots \wedge e^{i_p}$ and $\mu = g\, e^{i_1} \wedge \dots \wedge e^{i_p}$.

Then we have $\langle \omega, \mu \rangle = f \cdot g \langle e^{i_1} \wedge \dots \wedge e^{i_p}, e^{i_1} \wedge \dots \wedge e^{i_p} \rangle = f \cdot g \cdot \varepsilon(i_1) \cdot \dots \cdot \varepsilon(i_p)$.

With excersice 63, we write $\langle \omega, \mu \rangle \mathrm{vol} = f \cdot g\, \varepsilon(i_1) \cdot \dots \cdot \varepsilon(i_p)\, e^1 \wedge \dots \wedge e^n$.

The left handside gives

$$\begin{aligned} \omega \wedge * \mu &= f\, e^{i_1} \wedge \dots \wedge e^{i_p} \wedge *(g\, e^{i_1} \wedge \dots \wedge e^{i_p}) \\ &= f \cdot g\, e^{i_1} \wedge \dots \wedge e^{i_p} \wedge (\pm\, e^{i_{p+1}} \wedge \dots \wedge e^{i_n}) \\ &= f \cdot g\, \mathrm{sign}(i_1, \dots, i_n)\, \varepsilon(i_1) \dots \varepsilon(i_p)\, e^{i_1} \wedge \dots \wedge e^{i_n} \\ &= f \cdot g\, \mathrm{sign}^2(i_1, \dots, i_n)\, \varepsilon(i_1) \cdot \dots \cdot \varepsilon(i_p)\, e^1 \wedge \dots \wedge e^n \\ &= f \cdot g\, \varepsilon(i_1) \cdot \dots \cdot \varepsilon(i_p)\, e^1 \wedge \dots \wedge e^n . \end{aligned}$$

$$\Rightarrow \quad \omega \wedge * \mu = \langle \omega, \mu \rangle \mathrm{vol} .$$

Excersice 65:

When $w = \omega_x dx + \omega_y dy + \omega_z dz$, then

$$\begin{aligned} * dw &= *\left(\partial_y \omega_x\, dy \wedge dx + \partial_z \omega_x\, dz \wedge dx + \partial_x \omega_y\, dx \wedge dy + \partial_z \omega_y\, dz \wedge dy + \partial_x \omega_z\, dx \wedge dz + \partial_y \omega_z\, dy \wedge dz \right) \\ &= *\left((\partial_y \omega_z - \partial_z \omega_y)\, dy \wedge dz + (\partial_z \omega_x - \partial_x \omega_z)\, dz \wedge dx + (\partial_x \omega_y - \partial_y \omega_x)\, dx \wedge dy \right) \\ &= (\partial_y \omega_z - \partial_z \omega_y)\, dx + (\partial_z \omega_x - \partial_x \omega_z)\, dy + (\partial_x \omega_y - \partial_y \omega_x)\, dz \end{aligned}$$

Excersice 66:

$$\begin{aligned} * d * \omega &= * d\left(\omega_x\, dy \wedge dz + \omega_y\, dz \wedge dx + \omega_z\, dx \wedge dy \right) \\ &= *\left(\partial_x \omega_x\, dx \wedge dy \wedge dz + \partial_y \omega_y\, dy \wedge dz \wedge dx + \partial_z \omega_z\, dz \wedge dx \wedge dy \right) \\ &= (\partial_x \omega_x + \partial_y \omega_y + \partial_z \omega_z)\, *(dx \wedge dy \wedge dz) \\ &= \partial_x \omega_x + \partial_y \omega_y + \partial_z \omega_z \end{aligned}$$

Excersice 67:

$* 1 = dt \wedge dx \wedge dy \wedge dz$

$* dt \wedge dx \wedge dy \wedge dz = -1$

$* dt \wedge dx \wedge dy = -dz$

$* dt \wedge dy \wedge dz = -dx$

$* dt \wedge dx \wedge dz = dy$

$* dx \wedge dy \wedge dz = -dt$

$* dx \wedge dy = dt \wedge dz$

$*^2 = (-1)^{p(4-p)+1}$ in Minkowski $\mathbb{R}^4$

Look at next excersice!

Excersice 68 :

M oriented Semi-Riemannian manifold of dimension n and signature $\left(\overset{\overset{\#+}{\downarrow}}{n-s}, \overset{\overset{\#-}{\downarrow}}{s}\right)$.

On p-forms we have :

$$*^2(e^{i_1}\wedge\dots\wedge e^{i_p}) = \mathrm{sign}(i_1,\dots,i_n)\,\varepsilon(i_1)\dots\varepsilon(i_p)\,*(e^{i_{p+1}}\wedge\dots\wedge e^{i_n})$$
$$= \mathrm{sign}^2(i_1,\dots,i_n)\,\varepsilon(i_1)\cdot\ldots\cdot\varepsilon(i_n)\cdot(-1)^{p(n-p)}\;e^{i_1}\wedge\dots\wedge e^{i_p}$$
$$= (-1)^{\#-}\cdot(-1)^{p(n-p)}\;e^{i_1}\wedge\dots\wedge e^{i_p}$$

$$\Rightarrow \quad *^2 = (-1)^{p(n-p)+s}.$$

Excersice 69 :

Define $\quad \varepsilon_{i_1\dots i_n} = \begin{cases} \mathrm{sign}(i_1,\dots,i_n) & \text{, all } i_j \text{ distinct} \\ 0 & \text{, otherwise} \end{cases}$

We have $\quad \varepsilon^{i_1\dots i_p}{}_{j_1\dots j_{n-p}} = g^{i_1k_1}\dots g^{i_pk_p}\,\varepsilon_{k_1\dots k_p j_1\dots j_{n-p}}$

$$= \varepsilon(i_1)\dots\varepsilon(i_p)\,\varepsilon_{i_1\dots i_p j_1\dots j_{n-p}}$$
$$= \begin{cases} \varepsilon(i_1)\dots\varepsilon(i_p)\,\mathrm{sign}(i_1,\dots,i_p,j_1,\dots,j_{n-p}) & \text{, if } \{j_1,\dots,j_{n-p}\} = \{i_{p+1},\dots,i_n\} \\ 0 & \text{, otherwise} \end{cases}$$

$$*\omega = \frac{1}{p!}\,\omega_{i_1\dots i_p}\,*(e^{i_1}\wedge\dots\wedge e^{i_p})$$
$$= \frac{1}{p!}\,\omega_{i_1\dots i_p}\,\mathrm{sign}(i_1,\dots,i_p,j_1,\dots,j_{n-p})\,\varepsilon(i_1)\dots\varepsilon(i_p)\;e^{j_1}\wedge\dots\wedge e^{j_{n-p}}$$
$$= \frac{1}{p!\,(n-p)!}\,\omega_{i_1\dots i_p}\,\varepsilon^{i_1\dots i_p}{}_{j_1\dots j_{n-p}}\;e^{j_1}\wedge\dots\wedge e^{j_{n-p}}$$

$$\Rightarrow \quad (*\omega)_{j_1\dots j_{n-p}} = \frac{1}{p!}\,\varepsilon^{i_1\dots i_p}{}_{j_1\dots j_{n-p}}\,\omega_{i_1\dots i_p}.$$

Excersice 70 :

$$*_s d_s *_s E = *_s d_s\left(E_x\,dy\wedge dz + E_y\,dz\wedge dx + E_z\,dx\wedge dy\right)$$
$$= *_s\left(\partial_x E_x\,dx\wedge dy\wedge dz + \partial_y E_y\,dy\wedge dz\wedge dx + \partial_z E_z\,dz\wedge dx\wedge dy\right)$$
$$= \partial_x E_x + \partial_y E_y + \partial_z E_z$$
$$= \vec{\nabla}\cdot\vec{E}$$

$$*_s d_s *_s B = *_s d_s\left(B_x\,dx + B_y\,dy + B_z\,dz\right)$$
$$= (\partial_y B_z - \partial_z B_y)\,dx + (\partial_z B_x - \partial_x B_z)\,dy + (\partial_x B_y - \partial_y B_x)\,dz$$
$$= \vec{\nabla}\times\vec{B}$$

So $\quad \begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho \\ -\frac{\partial\vec{E}}{\partial t} + \vec{\nabla}\times\vec{B} &= \vec{j} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} *_s d_s *_s E &= \rho \\ -\partial_t E + *_s d_s *_s B &= j \end{aligned}$

Excersice 71 :

If $F = B + E \wedge dt$, we have

$$*F = *(B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy) + *(E_x\, dx \wedge dt + E_y\, dy \wedge dt + E_z\, dz \wedge dt)$$
$$= - *_s B \wedge dt + *_s E$$

$$d*F = d(*_s E - *_s B \wedge dt) = *_s \partial_t E \wedge dt + d_s *_s E - d_s(*_s B \wedge dt)$$
$$= *_s \partial_t E \wedge dt + d_s *_s E - d_s *_s B \wedge dt$$

$$*d*F = *(*_s \partial_t E \wedge dt + d_s *_s E - d_s *_s B \wedge dt)$$
$$= -*_s^2 \partial_t E - *_s d_s *_s E \wedge dt + *_s d_s *_s B$$
$$= \underbrace{-\partial_t E + *_s d_s *_s B}_{=j} - \underbrace{*_s d_s *_s E}_{=\rho} \wedge dt \qquad \Big\} \; *_s^2 \partial_t E = (-1)^{1(3-1)+0} \partial_t E = \partial_t E$$
$$= j - \rho\, dt$$
$$= J \; .$$

Excersice 72 :

If we take $F_\pm = \frac{1}{2}(F \pm *F)$, we have in Riemannian case $(*^2 = 1)$:

- $F_+ + F_- = \frac{1}{2}(F + *F) + \frac{1}{2}(F - *F) = F$.
- $*F_\pm = \frac{1}{2}(*F \pm \underset{=1}{*^2} F) = \frac{1}{2}(*F \pm F) = \pm \frac{1}{2}(F \pm *F) = \pm F_\pm$.

Excersice 73 :

In the Lorentzian case $(*^2 = -1)$, we should write $F_\pm = \frac{1}{2}(F \mp i \cdot *F)$, then

- $F_+ + F_- = \frac{1}{2}(F - i \cdot *F) + \frac{1}{2}(F + i \cdot *F) = F$.
- $*F_\pm = \frac{1}{2}(*F \mp i \cdot \underset{=-1}{*^2} F) = \frac{1}{2}(*F \pm iF) = \frac{1}{2} i(-i*F \pm F) = \pm i \cdot \frac{1}{2}(F \mp i \cdot *F)$
$= \pm i F_\pm$.

Excersice 74 :

$$*_s E = iB \iff *_s^2 E = i *_s B \iff E = i *_s B \iff *_s B = -iE .$$

If at every time t we have $B = -i(E_1\, dx^2 \wedge dx^3 + E_2\, dx^3 \wedge dx^1 + E_3\, dx^1 \wedge dx^2)$, both equations hold because :

$$*_s B = -i(E_1\, dx^1 + E_2\, dx^2 + E_3\, dx^3) = -iE \; ,$$

and they are equivalent.

Excersice 75 :

Starting from $\partial_t B + d_s E = 0$ we have $\partial_t B = ik_0 B$ and $d_s E = -E \wedge d_s e^{ik_\mu x^\mu}$
$= -iE \wedge {}^3k$

$\Rightarrow \quad ik_0 B = +iE \wedge {}^3k \quad \Leftrightarrow \quad -{}^3k \wedge E = k_0 B$.

Excersice 76 :

$${}^3k \wedge E = -ik_0 *_s E \quad \Leftrightarrow \quad 0 = (ik_0E_1 + k_2E_3 - k_3E_2)dx^2 \wedge dx^3 + (ik_0E_2 + k_3E_1 - k_1E_3)dx^3 \wedge dx^1 + (ik_0E_3 + k_1E_2 - k_2E_1)dx^1 \wedge dx^2$$

$$\Leftrightarrow \quad 0 = \begin{pmatrix} ik_0 & -k_3 & k_2 \\ k_3 & ik_0 & -k_1 \\ -k_2 & k_1 & ik_0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

This matrix is antisymmetric and has odd dimension $\Rightarrow$ its determinant is zero.

From $\det \begin{pmatrix} ik_0 & -k_3 & k_2 \\ k_3 & ik_0 & -k_1 \\ -k_2 & k_1 & ik_0 \end{pmatrix} = 0$ it follows $-k_0^2 + k_1^2 + k_2^2 + k_3^2 = 0 \quad \Leftrightarrow \quad k_\mu k^\mu = 0$.

Excersice 77 :

Choosing $k = dt - dx$ and $E = dy - idz$, we get : $k_\mu = (1,-1,0,0)$, $E_\mu = (0,0,1,-i)$

$$\Rightarrow \vec{E} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} e^{ik_\mu x^\mu} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} e^{i(t-x)}.$$

And $B_\mu = -i *_s E_\mu = (0,0,-i,-1)$

$$\Rightarrow \vec{B} = \begin{pmatrix} 0 \\ -i \\ -1 \end{pmatrix} e^{ik_\mu x^\mu} = \begin{pmatrix} 0 \\ -i \\ -1 \end{pmatrix} e^{i(t-x)}.$$

Excersice 78 :

For the self-dual case, we have seen that all solutions are left circularly polarized.
Now take the anti-self-dual case and develop the same steps as in the book. We have now :

$*_s E = -iB \quad , \quad *_s B = iE$

$\Rightarrow B \wedge {}^3k = 0$ and $\langle E, {}^3k \rangle = 0$ still holds

But this time we obtain ${}^3k \wedge E = ik_0 *_s E$, so we have also $k_\mu k^\mu = 0$

$\Rightarrow$ choose $k = dt - dx$ and $E = a\,dy + b\,dz$, since $\langle {}^3k, E \rangle = 0$.

To hold the above equation, we must have $a = -ib \rightarrow E = -idy + dz$, which is a right circularly polarized wave.

Excersice 79 :

If F is self-dual, then the pullback $p^*F = p^*B + p^*(E \wedge dt) = B - E \wedge dt$
$\Leftrightarrow \quad *_s E = iB$

$$*(p^*F) = -*_s B \wedge dt - *_s E = -(-iE) \wedge dt - iB = -i(B - E \wedge dt) = -ip^*F$$

$\Rightarrow p^*F$ is anti-self-dual.

The vice versa follows from $p^*p^*F = F$.

Remark on excersice 76 :

We can show from ${}^3k \wedge E = -ik_0 *_s E$ that $k_\mu k^\mu = 0$ by taking the inner product of both sides:
Since we are complex now, the inner product has to be sesquilinear :

$$\bullet \langle {}^3k \wedge E, {}^3k \wedge E \rangle = \langle k_i E_j dx^i \wedge dx^j, k_e E_m dx^e \wedge dx^m \rangle = k_i E_j k_e^* E_m^* \langle dx^i \wedge dx^j, dx^e \wedge dx^m \rangle$$
$$= k_i E_j k_e^* E_m^* (\delta^{ie}\delta^{jm} - \delta^{im}\delta^{je})$$
$$= \langle {}^3k, {}^3k \rangle \langle E, E \rangle - \langle {}^3k, E \rangle^2$$
$$\big\downarrow \; \langle {}^3k, E \rangle = 0$$
$$= \langle {}^3k, {}^3k \rangle \langle E, E \rangle$$

$$\bullet \langle -ik_0 *_s E, -ik_0 *_s E \rangle = -ik_0 ik_0 \langle *_s E, *_s E \rangle = k_0^2 \langle E, E \rangle$$

Since the forms are equal, their inner product must be the same $\Rightarrow \langle {}^3k, {}^3k \rangle = k_0^2$

$\Rightarrow \quad -k_0^2 + k_1^2 + k_2^2 + k_3^2 = 0 . \quad \Leftrightarrow \quad k_\mu k^\mu = 0$ with Minkowski metric.

Excersice 80 :

At first we change coordinates to polar coordinates to simplify the calculations.

$$\left.\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}\right\} \quad E = \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{r\cos\varphi(\sin\varphi\, dr + r\cos\varphi\, d\varphi) - r\sin\varphi(\cos\varphi\, dr - r\sin\varphi\, d\varphi)}{r^2}$$
$$= \cos^2\varphi\, d\varphi + \sin^2\varphi\, d\varphi$$
$$= d\varphi$$

$\Rightarrow dE = d^2\varphi = 0 .$

Expressing path γ_0 and γ_1 in polar coordinates, we have $\gamma_0 : [0,\pi] \to S, \; t \mapsto (1, \pi - t)$
$\gamma_1 : [0,\pi] \to S \; ; \; t \mapsto (1, \pi + t)$.

1) $\int_{\gamma_0} E = \int_0^\pi E_\mu(\gamma_0(t))\, \partial_t \gamma_0^\mu(t)\, dt = \int_0^\pi 1 \cdot (-1)\, dt = -\pi .$

2) $\int_{\gamma_1} E = \int_0^\pi E_\mu(\gamma_1(t))\, \partial_t \gamma_1^\mu(t)\, dt = \int_0^\pi 1 \cdot 1\, dt = \pi .$

Excersice 81 :

Let $\gamma_0(t)$ and $\gamma_1(t)$ be two paths between arbitrary points $p, q \in \mathbb{R}^n$.

$\gamma : [0,1] \times [0,T] \to \mathbb{R}^n, \quad (s,t) \mapsto \gamma(s,t) = (1-s)\gamma_0(t) + s \cdot \gamma_1(t) .$

This function is a smooth function with $\gamma(0,t) = \gamma_0(t)$ and $\gamma(1,t) = \gamma_1(t) \Rightarrow$ homotopy between γ_0 and γ_1 !

Since γ_0 and γ_1 can be choosen arbitrarely, any two paths between the points $p, q \in \mathbb{R}^n$ are homotopic $\Rightarrow \mathbb{R}^n$ is simply connected.

Excersice 82 :

„$\Rightarrow$" Let $E = d\varphi$ be an exact form and γ a loop starting and ending at $p \in M$.

$$\oint_\gamma E = \int_0^1 d\varphi(\gamma'(t))\, dt = \int_0^1 \gamma'(t)(\varphi)\, dt = \int_0^1 \frac{d}{ds}\varphi(\gamma(s))\Big|_{s=t} dt = \int_0^1 (\varphi(\gamma(t)))'\, dt = \varphi(\overset{p}{\overset{\shortparallel}{\gamma(1)}}) - \varphi(\overset{p}{\overset{\shortparallel}{\gamma(0)}}) = 0$$

„$\Leftarrow$" Suppose E is not exact. Then M is not simply connected $\Rightarrow \exists \gamma, \gamma'$ with no homotopy between them. We must be able to find $x, y \in M$ and paths γ, γ' with $\int_\gamma E \neq \int_{\gamma'} E$ or else we could use this integral to define φ with $E = d\varphi$. Defining $\tilde{\gamma}$ to be the path taking γ forward and γ' backwards, we found a path with $\oint_{\tilde\gamma} E \neq 0$ ↯

Excersice 83 :

Choosing coordinates (t, x^μ) on $S^1 \times M$, consider the 1-form $\omega = dt$.
Clearly $d\omega = 0$, so the form is closed.
But there is no continous function φ defined on the whole manifold such that $\omega = d\varphi$, since if we try to define it using $\int_\gamma \omega$, where γ is a path around S^1 and fixed on M, we obtain $\varphi(t, x^\mu) = t$, which depends on the winding number of γ.
So we can have $\int_\gamma \omega \neq 0$ and therefore ω isn't exact $\Rightarrow$ $S^1 \times M$ is not simply connected.

Excersice 84 :

Consider the open subsets $U_i^+ \subset D^n$, where $U_i^+ = \{x \in D^n : x_i > 0\}$.
Let $\phi : U_i^+ \to \mathbb{R}^n$ map $(x_1, \dots, x_n)$ to $\frac{\|x\|}{x_i}(x_1, \dots, x_i, \dots, x_n)$.
The image of ϕ is $\{x \in \mathbb{R}^n : 0 < x_i \leq 1\}$, with points on the boundary $(\|x\| = 1)$ mapping to points with $x_i = 1$.
We can compose with another mapping $\mathbb{R}^n \to \mathbb{R}^n$ that maps x_i to $(1 - x_i)$.
This gives us a chart $\varphi : U_i^+ \to H^n$ for the boundary points on $x_i = 0$.
Finally, charts of the form U_i^+ and U_i^- cover D^n, hence forming an atlas.

Excersice 85 :

Tangent vectors at $p \in M$ were defined to be maps from $C^\infty(M)$ to $\mathbb{R}$ obeying linearity and the Leibniz law. If we map the points on the boundary to the boundary of H^n, where $x^n \geq 0$ is the special coordinate, we will still have all the derivatives in the other directions.
The derivative in the x^n direction still works as well, since the functions must be smooth on $-\varepsilon < x^n$ too!

Excersice 86 :

Let $\{U_\alpha\}$ be the original atlas with $\{f_\alpha\}$ the corresponding partition of unity.
Let $\{V_\beta\}$ be another atlas where all the charts have the same orientation as in the original atlas, with $\{g_\beta\}$ $(\mathrm{supp}(g_\beta) \subset V_\beta)$ partition of unity.

Then $\sum_\alpha \int_{U_\alpha} f_\alpha \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \left(\sum_\beta g_\beta \omega\right) = \sum_\alpha \sum_\beta \int_{U_\alpha \cap V_\beta} f_\alpha g_\beta \omega = \sum_\beta \int_{V_\beta} g_\beta \left(\sum_\alpha f_\alpha \omega\right) = \sum_\beta \int_{V_\beta} g_\beta \omega$.

Interchanging summation and integration is allowed because we have finite sums.
So $\int_M \omega$ is independent of the choice of charts and partition of unity.

Excersice 87 :

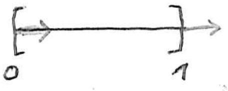
Using the same charts as defined in ex. 84, we see that $\partial D^n = \{x \in D^n : x_1^2 + \dots + x_n^2 = 1\}$.
This is the same set of points that can be used to define S^{n-1}.

Ich liebe dich ♡

Excersice 88 :

With $\omega = f(x)$ we have $\int_M d\omega = \int_{\partial M} \omega \iff \int_0^1 f'(x)\,dx = f(1) - f(0)$.

Because $\partial M = \{0,1\}$ and dx defines the orientation by defining increasing x to be positive at 1 and negative at 0.



Excersice 89 :

choose the function $f'(x) = 1 \Rightarrow f(x) = x$.
This integral would clearly diverge, whereas by Stokes' theorem it would be zero.

Excersice 90 :

Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas of M.

Let $V_\alpha = S \cap U_\alpha$ and $\psi_\alpha = \varphi_\alpha|_{V_\alpha}$, then V_α are open subsets of S and the functions $\psi_\beta \circ \psi_\alpha^{-1}$ from $\mathbb{R}^k$ to $\mathbb{R}^k$ are smooth because φ_α are smooth.

Hence $\{(V_\alpha, \psi_\alpha)\}$ forms an atlas of S.

Excersice 91 :

As a topological space, it is compact since it is closed (beeing pre-image of the closed set $\{1\}$ under the norm function) and bounded (all points have norm less than 2).
Define the same atlas as used in excersice 84 $\{(U_\alpha^{+/-}; \varphi_\alpha^{+/-})\}$.
(U_i^+, φ_i^+) is a chart that maps $S^{n-1} \cap U_i^+$ bijectively to the hyperplane $\{x \in \mathbb{R}^n, x_n = 1\}$ which is $\mathbb{R}^{n-1}$.

Excersice 92 :

Let $V \subset M$ be an open set and let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas for M.
Then the family of open sets $V \cap U_\alpha$ together with the charts $\varphi_\alpha|_{V\cap U_\alpha}$ forms a suitable an atlas of V.

Excersice 93 :

The same as excersice 90, just with boundary (replace $\mathbb{R}^k$ with H^k).
Is S a submanifold with boundary $(S \cap U = \varphi^{-1} H^k)$, than the boundary ∂S is just the pre-image of $\{(x^1, \dots, x^k) : x^k = 0\}$ which is a $(k-1)$-dimensional hyperplane of $\mathbb{R}^k$.
Thus $\partial S \cap U = \varphi^{-1} \mathbb{R}^{k-1}$ and ∂S is a $(k-1)$-dimensional submanifold of M.

Excersice 94 :

use the same chart as in excersice 84. Use the induced topology of $\mathbb{R}^n$: $D^n \cap U = \varphi^{-1} \mathbb{R}^n$ and for points on the boundary $D^n \cap U = \varphi^{-1} H^n$.

Excersice 95 :

Let $\omega = \omega_x dx + \omega_y dy$, then $d\omega = (\partial_x \omega_y - \partial_y \omega_x)\, dx \wedge dy$

So $\int_S (\partial_x \omega_y - \partial_y \omega_x)\, dx\, dy = \int_{\partial S} (\omega_x dx + \omega_y dy)$

Excersice 96 :

The usual Stokes law is $\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$ for a vector field $\vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$.

Let $\omega = F_x \, dx + F_y \, dy + F_z \, dz$. Given the orientation of S, we can say that the normal dS points in the z-direction. Thus $(\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$ corresponds to the z-componet of $d\omega$, which is the same as in usual Stokes law. Also ω reduces to $F_x \, dx + F_y \, dy$ (orthogonal to z) which is the same as $\vec{F} \cdot d\vec{r}$.

Excersice 97 :

Let $\omega = \omega_x \, dy \wedge dz + \omega_y \, dz \wedge dx + \omega_z \, dx \wedge dy$.

We have shown that in $\mathbb{R}^3$ we then have $d\omega = \operatorname{div} \vec{\omega} \, dx \wedge dy \wedge dz$.

To show that integrating the normal of $\vec{\omega}$ over the surface is the same as integrating ω over the surface, choose local coordinates such that the surface lies in the plane $z=0$. Then the normal component of $\vec{\omega}$ is just ω_z and restricted to the surface, ω becomes $\omega_z \, dx \wedge dy$ and integrating the two gives the same result: $\int_V (\vec{\nabla} \cdot \vec{\omega}) \, dV = \int_{\partial V} \vec{\omega} \cdot d\vec{s}$

Excersice 98 :

Let φ be a map from M to N and let ω be a p-form on N.

- Suppose ω is closed ($d\omega = 0$):

$$d(\varphi^* \omega) = \varphi^* d\omega = \varphi^* 0 = 0.$$

- Suppose ω is exact ($\exists (p-1)$-form μ on N with $\omega = d\mu$):

$$\varphi^* \omega = \varphi^* d\mu = d(\varphi^* \mu).$$

So if ω is closed/exact then $\varphi^* \omega$ is also closed/exact.

Excersice 99 :

Let $\phi : M \to M'$ be a map, then we showed that closed forms stay closed and exact forms stay exact under pullback.

We define $\phi^* : H^p(M') \to H^p(M)$, $[\omega] \mapsto [\phi^* \omega]$.

It is clearly linear because of the properties of the pullback ϕ^*. It is well-defined:

Let $\omega' \in [\omega] \Rightarrow \exists \mu \;\; \omega - \omega' = d\mu$, so $\phi^* \omega' \in [\phi^* \omega]$ because $\phi^* \omega - \phi^* \omega' = d(\phi^* \mu)$.

If $\psi : M' \to M''$ is another map, then $(\psi \phi)^* : H^p(M'') \to H^p(M)$.

And $(\psi \phi)^* = \phi^* \psi^*$ because of the properties of the pullback.

Excersice 100 :

Define $r^2 = x^2 + y^2$, then $r \, d\theta = x \, dy - y \, dx$.

$$* dz = dx \wedge dy = d(r \cos\theta) \wedge d(r \sin\theta) = (\cos\theta \, dr - r \sin\theta \, d\theta) \wedge (\sin\theta \, dr + r \cos\theta \, d\theta) = r \, dr \wedge d\theta.$$

Excersice 101 :

$$* d\theta = \frac{x * dy - y * dx}{x^2 + y^2} = \frac{x\, dz \wedge dx - y\, dy \wedge dz}{x^2+y^2} = \frac{(r\cos\theta)\, dz \wedge d(r\cos\theta) - (r\sin\theta)\, d(r\sin\theta) \wedge dz}{r^2}$$

$$= \frac{1}{r}\left(\cos\theta\, dz \wedge (\cos\theta\, dr - r\sin\theta\, d\theta) - \sin\theta\,(\sin\theta\, dr + r\cos\theta\, d\theta) \wedge dz \right)$$

$$= \frac{1}{r}\left(\cos^2\theta\, dz \wedge dr + \sin^2\theta\, dz \wedge dr \right)$$

$$= \frac{1}{r}\, dz \wedge dr .$$

Excersice 102 :

$$d * B = * j \iff d\big(g(r)\, d\theta \big) = f(r)\, r\, dr \wedge d\theta \iff g'(r)\, dr \wedge d\theta = f(r)\, r\, dr \wedge d\theta$$

$$\iff g'(r) = r \cdot f(r) .$$

Excersice 103 :

Define the map $p : S^1 \times S^{n-1} \to S^1 \ , \ (\theta, x^1, \ldots, x^{n-1}) \mapsto \theta$.

The 1-form $d\theta$ on S^1 is closed but not exact.

Then $\omega = p^* d\theta$ on $S^1 \times S^{n-1}$ is also closed but not exact.

Excersice 104 :

Let $M = \mathbb{R} \times S^2$ with metric $g = dr^2 + f(r)^2 \left(d\phi^2 + \sin^2\phi\, d\theta^2 \right)$.

Let $E = e(r)\, dr$.

$$\Rightarrow dE = e'(r)\, dr \wedge dr + 0 \cdot d\phi \wedge dr + 0 \cdot d\theta \wedge dr = 0 \qquad \forall\, e(r)$$

We have $* E = e(r) * dr = e(r)\, f(r)^2 \sin\phi\, d\phi \wedge d\theta$, because $vol = f(r)^2 \sin\phi\, dr \wedge d\phi \wedge d\theta$.

So $d * E = \partial_r \left(e(r)\, f(r)^2 \right) \sin\phi\, dr \wedge d\phi \wedge d\theta = 0$ only holds when $e(r)\, f(r)^2 = \text{const}$.

Since $f(r)^2$ should equal r^2 for large $|r|$, where the space is Euclidean, we now that the electric field looks like $E = \frac{q}{4\pi r^2}\, dr$, thus $e(r) = \frac{q}{4\pi r^2} \iff e(r)\, r^2 = \frac{q}{4\pi}$.

So we choose the constant to be $q/4\pi \Rightarrow e(r) = \frac{q}{4\pi f(r)^2}$.

Excersice 105 :

Choose $\phi(r) = - \int_0^r e(s)\, ds$, then we have $E = -d\phi$.

Excersice 106 :

Let $E = \frac{q\, dr}{4\pi r^2}$, then $\int_{S^2} * E = \int_{S^2} \frac{q}{4\pi \cancel{r^2}} \cancel{r^2} \sin\phi\, d\theta \wedge d\phi = \frac{q}{4\pi} \int_0^{2\pi}\!\!\int_0^{\pi} \sin\phi\, d\theta\, d\phi = \frac{q}{\cancel{4\pi}} \cdot \cancel{4\pi} = q$.

Excersice 107 :

We calculated the above integral, where $r > 0$. Thus the orientation of increasing r points outward. In the case $r < 0$ the direction of increasing r is pointing inwards, so to remain the same orientation of the normal, we have to change the sign of the volume form. $\rightsquigarrow -q$.

Excersice 108 :

Let E be a 1-form in n-dimensional space .

Then it must have nonzero H^{n-1} in order for there to be a $(n-1)$-dimensional surface S with $\int_S *E \neq 0$, when $*E$ is closed .

Excersice 109 :

dw is the sum of three terms like $\partial_x \left(\frac{x}{(x^2+y^2+z^2)^{3/2}} \right) = \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}}$.

Thus $dw = \frac{3(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2}{(x^2+y^2+z^2)^{5/2}} dx \wedge dy \wedge dz = 0$.

Excersice 110 :

An $(n-1)$-form on $\mathbb{R}^n - \{0\}$ that is closed but not exact is :

$$\omega = \frac{x^1\, dx^2 \wedge \dots \wedge dx^n + x^2\, dx^3 \wedge \dots \wedge dx^n \wedge dx^1 + \dots + x^n\, dx^1 \wedge \dots \wedge dx^{n-1}}{\left((x^1)^2 + \dots + (x^n)^2 \right)^{n/2}}$$

$\Rightarrow H^{n-1}(\mathbb{R}^n - \{0\}) \neq 0$.

Excersice 111 :

$$B = * \frac{m\, dr}{4\pi f(r)^2} = \frac{m \cancel{f(r)^2} \sin\phi\, d\theta \wedge d\phi}{4\pi \cancel{f(r)^2}} = \frac{m}{4\pi} \sin\phi\, d\theta \wedge d\phi$$

So $\int_{S^2} B = \frac{m}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sin\phi\, d\theta\, d\phi = \frac{m}{\cancel{4\pi}} \cdot \cancel{4\pi} = m$.

In ordinary space $\mathbb{R}^3$, $dB = 0$ would imply that B is exact and hence

$\int_{S^2} B = \int_{S^2} dA = \int_{\partial S^2} A = 0$, since S^2 has no boundary .

Part II

Excersice 1 :

$$T = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in SO(3,1)$$

Because

$$\begin{aligned} g(Tv, Tw) &= -(\cosh\phi\, v^1 - \sinh\phi\, v^2)(\cosh\phi\, w^1 - \sinh\phi\, w^2) + (\cosh\phi\, v^2 - \sinh\phi\, v^1)(\cosh\phi\, w^2 - \sinh\phi \\ &\quad + v^3 w^3 + v^4 w^4 \\ &= -\cosh^2\phi\, v^1 w^1 - \sinh\phi\, v^2 w^2 + \cosh^2\phi\, v^2 w^2 + \sinh^2\phi\, v^1 w^1 + v^3 w^3 + v^4 w^4 \\ &= -v^1 w^1 + v^2 w^2 + v^3 w^3 + v^4 w^4 \\ &= g(v, w) \end{aligned}$$

and $\det(T) = \cosh\phi \cdot \begin{vmatrix} \cosh\phi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \sinh\phi \cdot \begin{vmatrix} -\sinh\phi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cosh^2\phi - \sinh^2\phi = 1$.

Analog for the other transformations .

Excersice 2 :

$$\left.\begin{array}{l} P : (t,x,y,z) \mapsto (t,-x,-y,-z) \\ T : (t,x,y,z) \mapsto (-t,x,y,z) \end{array}\right\} \notin SO(3,1) \quad \text{because} \quad \det(P) = \det(T) = -1 .$$

But they lie in $O(3,1)$ because the obey $g(Tv, Tw) = g(v,w)$.

The product $PT : (t,x,y,z) \mapsto (-t,-x,-y,-z)$ lies in $O(3,1)$ and has $\det(PT) = 1$, so it lies in $SO(3,1)$.

Excersice 3 :

$SU(n)$ is a matrix group :

1) closed under matrix multiplication :

Let $U, V \in SU(n)$. We have $V \cdot x \in \mathbb{C}^n$ and since $g(Ux, Uy) = g(x,y) \;\; \forall x,y \in \mathbb{C}^n$

We have $g(UVx, UVy) = g(Vx, Vy) = g(x,y) \Rightarrow U \cdot V \in U(n)$.

And $\det(U \cdot V) = \det(U) \cdot \det(V) = 1 \Rightarrow SU(n) \ni U \cdot V$.

2) Inverses :

Since $\det(U) = 1 \neq 0$ for all $U \in SU(n)$, there exists an inverse $\forall U \in SU(n)$.

3) Identity :

Let $U \in SU(n) \Rightarrow \mathbb{1}_{n\times n} \cdot U = U \cdot \mathbb{1}_{n \times n} = U$.

Excersice 4 :

- Product and inverse are smooth maps :

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices, then the ij'th entry of the product is $\sum_k a_{ik} b_{kj}$ which is smooth as a function from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

The inverse is shown to be smooth by the explicit formula using the adjugate matrix.

- The groups are submanifolds :

$GL(n,\mathbb{R})$ is a submanifold since it is an open subset, the pre-image of $\mathbb{R}\setminus\{0\}$ under det.

For the other groups just use the theorem of regular value.

Excersice 5 :

Given a Lie group $\mathcal{G}$, the identity component $\mathcal{G}_0$ is the connected component containing the identity.

- Closed under multiplication :

Suppose we have a path from the identity to $g \in \mathcal{G}_0$. Map this path to a new path by multiplying each element by $h \in \mathcal{G}_0$. This path starts at h and ends at hg and since the mapping is continous, must remain in $\mathcal{G}_0 \Rightarrow g \cdot h \in \mathcal{G}_0 \;\; \forall h, g \in \mathcal{G}_0$.

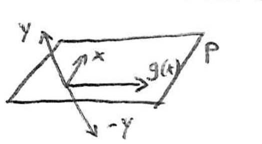
- Inverse :

Let $i : \mathcal{G} \to \mathcal{G}$ be the inverse function, which is continuous since $\mathcal{G}$ is a Lie group. Combine i with the path to $g \in \mathcal{G}_0$ then we get a path from the identity to g^{-1}, which is continuous $\Rightarrow g^{-1} \in \mathcal{G}_0$.

$\Rightarrow \mathcal{G}_0 \subseteq \mathcal{G}$ subgroup, restricting product and inverse to $\mathcal{G}_0$ we get that $\mathcal{G}_0$ is a Lie group.

Excersice 6 :

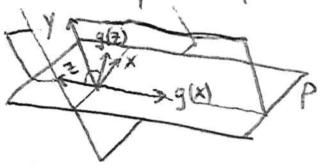
First, every element of $O(3)$ preserves the length of vectors and the angles between them. Let $x \in \mathbb{R}^3$ and $g \in O(3)$. Consider the plane P spanned by x and $g(x)$.



There are two vectors that are orthogonal to P, call it y and $-y$.

Since y is orthogonal to x, $g(y)$ must be orthogonal to $g(x)$. Likewise $y \perp g(x) \Rightarrow g(y) \perp x$.
In other words $g(y)$ is orthogonal to the same plane P, so it must equal either y or $-y$.

- If $g(y) = y$, then g is a rotation about the axis spanned by y



Because: Consider a vector z in the plane spanned by x and y.
Since the angles must be preserved, $g(z)$ must lie in the plane spanned by $g(x)$ and y, at the same position that rotation around the y axis would leave it.
Then consider a vector z' in the x-$g(x)$-plane and use again preservation of angles to show that $g(z)$ lies where it should. These two facts imply the claim.

- If $g(y) = -y$, then we can compose with a reflection through the plane P and it follows from the above that the composition is a rotation.

This completes the proof that g is a rotation possibly combined with a reflection.

If g is just a rotation (through an angle θ) then we can construct a path γ in $O(3)$ such that $\gamma(t)$ is a rotation (through $t\theta$). This shows that g is in the identity component.
$\Rightarrow$ Idetity Componet of $O(3)$ is $SO(3)$.

The rotations with reflections are not in the idetity component because the determinant function is continous and has image $\{1, -1\}$, thus divides $O(3)$ into two disjoint subsets.
Using the fact that reflections have determinant -1 and that if $h \in O(3)$ is a rotation, then gh are in the same connected component of $O(3)$.

Moreover reflections aren't really a subgroup since they're not closed under matrix multiplication

Excersice 7 :

Consider the vector $u \in \mathbb{R}^4$ with $u = (1, 0, 0, 0)$. Let $A \in SO(3,1)$.
Then $Au = (a_{11}, a_{21}, a_{31}, a_{41})$ is the first column of A.
Since $\langle Au, Au \rangle = \langle u, u \rangle = -1$, we have $-a_{11}^2 + a_{21}^2 + a_{31}^2 + a_{41}^2 = -1$
$\Rightarrow a_{11}^2 \geq 1$.

For PT we have $a_{11} = -1$ and for the idetity $a_{11} = 1$, so there is no continous path from 1 to -1 such that $a_{11}^2 \geq 1$ $\Rightarrow$ $SO(3,1)$ has at least two connected components.
The connected component of the idetity is generated by elements such as in excersice 1, together with transformations from $SO(3)$ which leave the time unchanged.
We have two connected components of $SO(3,1)$ because its like in $O(3)$, we have spacetime-rotations and spacetime-reflections.

Excersice 8 :

There exists only one element with the properties of the identity and the inverse :

Let $1' \in \mathcal{G}$ be another identity : $1' \cdot g = g \cdot 1' = g$

Then $1' \cdot g = 1 \cdot g$ and $g \cdot 1' = g \cdot 1$ imply $1 = 1'$.

- $\left.\begin{aligned} \rho(g) &= \rho(1 \cdot g) = \rho(1) \cdot \rho(g) \\ &= \rho(g \cdot 1) = \rho(g) \cdot \rho(1) \end{aligned}\right\}$ these are the properties of the identity, so $\rho(1) = 1$
- $\left.\begin{aligned} 1 = \rho(1) &= \rho(g \cdot g^{-1}) = \rho(g) \cdot \rho(g^{-1}) \\ &= \rho(g^{-1} \cdot g) = \rho(g^{-1}) \cdot \rho(g) \end{aligned}\right\}$ properties of the inverse, so $\rho(g^{-1}) = \rho(g)^{-1}$

Excersice 9 :

$U(1)$ consists of all numbers that preserve the inner product on $\mathbb{C}$: $\langle v, w \rangle = \bar{v} \cdot w$.

So from $\langle uv, uw \rangle = \bar{u} \cdot \bar{v} \cdot uw = \bar{u} \cdot u \cdot \bar{v} \cdot w \overset{!}{=} \bar{v} \cdot w$ it follows that $\bar{u} \cdot u = |u|^2 = 1$

$\Rightarrow \quad U(1) = \{ u \in \mathbb{C} : |u|^2 = 1 \} = \{ e^{i\theta}, \theta \in \mathbb{R} \}$.

$U(1) \cong SO(2)$ with the isomorphism $\rho(e^{i\theta}) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

We have $\rho(e^{i\theta}) \rho(e^{i\phi}) = \begin{pmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} = \rho(e^{i(\theta+\phi)})$ $\Rightarrow$ ρ is homomorphism

The map is onto, since every element in $SO(2)$ is of the given form.

It is one-to-one, because $\rho(e^{i\theta}) \neq \rho(e^{i\phi})$ if $\theta, \phi \in (0, 2\pi)$ and $\phi \neq \theta$.

Excersice 10 :

$\mathcal{G} \times H = \{ (g,h) : g \in \mathcal{G}, h \in H \}$ is a group with product $(g,h)(g',h') = (gg', hh')$.

- Closed because $gg' \in \mathcal{G}$ and $hh' \in H$, so $(gg', hh') \in \mathcal{G} \times H$.
- Identity $(g,h) = 1 \cdot (g,h) = (1,1)(g,h) = (1 \cdot g, 1 \cdot h) = (g \cdot 1, h \cdot 1) = (g,h)(1,1) = (g,h) \cdot 1$
- Inverse $\left.\begin{aligned} 1 = (1,1) &= (gg^{-1}, hh^{-1}) = (g,h)(g^{-1}, h^{-1}) \\ &= (g^{-1}g, h^{-1}h) = (g^{-1}, h^{-1})(g,h) \end{aligned}\right\}$ $(g^{-1}, h^{-1}) = (g,h)^{-1}$.

If $\mathcal{G}$ and H are Lie groups, so is $\mathcal{G} \times H$. ← M, N manifolds $\Rightarrow$ $M \times N$ manifold and the maps are smooth.

$\mathcal{G} \times H$ is abelian $\Leftrightarrow$ $\mathcal{G}$ and H are abelian. ← straightforward calculation.

Excersice 11 :

$$\begin{aligned} (\rho \oplus \rho')(gh)(v,v') &= (\rho(gh)v, \rho'(gh)v') = (\rho(g)\rho(h)v, \rho'(g)\rho'(h)v') \\ &= (\rho(g), \rho'(g)) \cdot (\rho(h), \rho'(h)) \cdot (v,v') \\ &= (\rho \oplus \rho')(g) \cdot (\rho \oplus \rho')(h) \cdot (v,v') \end{aligned}$$

Excersice 12 :

Define $F : V \otimes V' \to W$ by $F(e_i \otimes e'_j) = f(e_i, e'_j)$.

This map is unique since f is fixed and is defined for a linearly independent basis.

So we have $f(v,v') = f(v^i e_i, v'^j e'_j) = v^i v'^j f(e_i, e'_j) = v^i v'^j F(e_i \otimes e'_j) = F(v \otimes v')$

Excersice 13 :

Well-definedness follows from $\rho(g)v \otimes \rho'(g)v' = v^i v'^j \rho(g) e_i \otimes \rho'(g) e'_j$, since every element of $V \otimes V'$ can be uniquely expressed as $v^i v'^j e_i \otimes e'_j$.

$$(\rho \otimes \rho')(gh)(v \otimes v') = \rho(gh)v \otimes \rho'(gh)v' = \rho(g)\rho(h)v \otimes \rho'(g)\rho'(h)v'$$
$$= \big(\rho(g) \otimes \rho'(g)\big)\big(\rho(h) \otimes \rho'(h)\big)(v \otimes v')$$
$$= (\rho \otimes \rho')(g) \cdot (\rho \otimes \rho')(h)\,(v \otimes v').$$

Excersice 14 :

$V \oplus \{0\}$ is an invariant subspace of $V \oplus V'$ because $(\rho \oplus \rho')(g)(v,0) = (\rho(g)v, 0) \in V \oplus \{0\}$.
Then we can define a subrepresentation of $\rho \oplus \rho'$ taking $v \oplus 0$ to $\rho(g)v \oplus 0$, but this is just the original representation ρ.
Analoge for ρ'.

Excersice 15 :

$$\rho_n(e^{i\theta} e^{i\phi})v = \rho_n(e^{i(\theta+\phi)})v = e^{in(\theta+\phi)}v = e^{in\theta} e^{in\phi} v = \rho_n(e^{i\theta})\rho_n(e^{i\phi})v.$$

Excersice 16 :

The only way to get a 1-dimensional representation of $U(1)$ is a reparametrization of θ in $\{ e^{i\theta} : \theta \in \mathbb{R} \}$. Let $\{ e^{i\phi} , \phi \in \mathbb{R} \}$ be another representation of $U(1)$.
Then there exists an isomorphism for one n from $e^{i\phi}$ to $e^{in\theta}$ such that they are the same.

Excersice 17 :

The tensor product $\rho_n \otimes \rho_m$ on $\mathbb{C} \otimes \mathbb{C}$ is given by

$$(\rho_n \otimes \rho_m)(e^{i\theta})(v \otimes v') = \rho_n(e^{i\theta})v \otimes \rho_m(e^{i\theta})v' = e^{in\theta} v \otimes e^{im\theta} v' = e^{i(n+m)\theta} v \otimes v'$$
$$= \rho_{n+m}(e^{i\theta})(v \otimes v').$$

Excersice 18 :

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a complex matrix , then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \begin{aligned} a &= \alpha + \delta \\ b &= \beta - i\gamma \\ c &= \beta + i\gamma \\ d &= \alpha - \delta \end{aligned}$$

There exists a unique solution $\alpha = \frac{1}{2}(a+d)$ $\gamma = \frac{1}{2i}(c-b)$
for every matrix A ! $\beta = \frac{1}{2}(b+c)$ $\delta = \frac{1}{2}(a-d)$

The matrix is hermitian $(A^+ = A)$ only if $\left.\begin{aligned} \bar{a} &= a \\ \bar{d} &= d \\ b &= \bar{c} \end{aligned}\right\} \alpha, \beta, \gamma, \delta \in \mathbb{R}$

The matrix is traceless $\Leftrightarrow$ $\alpha = 0$

Because $a + d = 0$ implies $\alpha = 0$!

Excersice 19 :

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \mathbb{1}$$

If (i,j,k) is a cyclic permutation of $(1,2,3)$ then

$$\left.\begin{aligned}
\sigma_1\sigma_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3 \\
\sigma_2\sigma_1 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3 \\
\sigma_2\sigma_3 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1 \\
\sigma_3\sigma_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_1 \\
\sigma_3\sigma_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 \\
\sigma_1\sigma_3 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2
\end{aligned}\right\} \quad \sigma_i\sigma_j = -\sigma_j\sigma_i = i\sigma_k$$

Excersice 20 :

Let $A = a + bI + cJ + dK = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} - bi\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - ci\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - di\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a-id & -c-ib \\ c-ib & a+id \end{pmatrix}$

$$\det(A) = \begin{vmatrix} a-id & -c-ib \\ c-ib & a+id \end{vmatrix} = (a-id)(a+id) - (c-ib)(-c-ib) = a^2 - (id)^2 - (-c^2 + (i\cdot b)^2)$$
$$= a^2 + b^2 + c^2 + d^2$$

$$A^{+}A = \begin{pmatrix} a+id & c+ib \\ -c+ib & a-id \end{pmatrix}\begin{pmatrix} a-id & -c-ib \\ c-ib & a+id \end{pmatrix} = \begin{pmatrix} (a+id)(a-id)+(c+ib)(c-ib) & (a+id)(-c-ib)+(c+ib)(a+id) \\ (-c+ib)(a-id)+(a-id)(c-ib) & (-c+ib)(-c-ib)+(a-id)(a+id) \end{pmatrix}$$

$$= \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}, \quad \text{if } a,b,c,d \in \mathbb{R}.$$

If $a^2+b^2+c^2+d^2 = 1 \Rightarrow A$ is unitary !

Moreover, since $\det(A) = a^2+b^2+c^2+d^2$, $\det(A)$ would be 1, so $A \in SU(2)$.

Excersice 21 :

For the spin-0 representation we use the space H_0 of polynomial functions homogeneous of degree 0, that is constant functions, $f(x,y) = c$.

Now for any $g \in SU(2)$, we have $(U_0(g)f)(v) = f(g^{-1}v) = c \Rightarrow U_0(g)f = f$.

$\Rightarrow U_0(g) = 1$.

Excersice 22 :

For the spin-$\frac{1}{2}$ representation $H_{1/2}$ is two dimensional and consits of functions homogeneous of degree one, that is $\{x,y\}$ form a basis. $\Rightarrow f(x,y) = ax + by = (a\ b)\cdot\begin{pmatrix} x \\ y \end{pmatrix}$

For $g \in SU(2)$, we have $g^{+}g = gg^{+} = \mathbb{1}$, so $g^{-1} = g^{+}$.

$$(U_{1/2}(g)f)(v) = f(g^{-1}v) = (a\ b)\, g^{-1}v = (a\ b)\, g^{+}v = \left(g\begin{pmatrix} a \\ b \end{pmatrix}\right)^{+} v = (gf)(v)$$

$\Rightarrow U_{1/2}(g) = g$.

Excersice 23 :

The dual given by $(\rho^*(g)f)(v) = f(\rho(g^{-1})v)$ is a representation, since

1) $(\rho^*(1)f)(v) = f(\rho(1)v) = f(v) \Rightarrow \rho^*(1) = 1$

2) $(\rho^*(gh)f)(v) = f(\rho((gh)^{-1})v) = f(\rho(h^{-1}g^{-1})v) = f(\rho(h^{-1})\rho(g^{-1})v) = (\rho^*(g)\rho^*(h)f)(v)$

$\Rightarrow \rho^*(gh) = \rho^*(g)\rho^*(h)$.

All the representations U_j of $SU(2)$ are equivalent to their duals.

Excersice 24 :

Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix commuting with all traceless hermitian matrices.

All of these are of the form $T = T^1\sigma_1 + T^2\sigma_2 + T^3\sigma_3$.

In order to commute $(ST = TS)$, it has to commute with the Pauli matrices!

$$\left.\begin{aligned} \cdot\ S\cdot\sigma_1 &= \begin{pmatrix} b & a \\ d & c \end{pmatrix} \\ \sigma_1\cdot S &= \begin{pmatrix} c & d \\ a & b \end{pmatrix} \end{aligned}\right\} \quad \begin{aligned} b &= c \\ a &= d \end{aligned}$$

$$\Rightarrow S = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\cdot\mathbb{1}.$$

$$\left.\begin{aligned} \cdot\ S\cdot\sigma_3 &= \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} \\ \sigma_3\cdot S &= \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} \end{aligned}\right\} \quad \left.\begin{aligned} b &= -b \\ c &= -c \end{aligned}\right\} \quad b = c = 0$$

Excersice 25 :

In the spin-1 representation $\mathcal{H}_1$ is 3 dimensional: $f(x,y) = ax^2 + bxy + cy^2 = (x\ y)\underbrace{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}}_{=T}\begin{pmatrix} x \\ y \end{pmatrix}$

Then
$$\begin{aligned}(U_1(g)f)(v) &= f(g^{-1}v) = (g^{-1}v)^\dagger T g^{-1}v \\ &= v^\dagger g T g^{-1} v \\ &= (gfg^{-1})(v).\end{aligned}$$

So $U_1(g)f = gfg^{-1}$ which was exactly the property of $\rho: SU(2) \to GL(3,\mathbb{R})$.

Since $GL(3,\mathbb{R}) \subset GL(3,\mathbb{C})$ is a subgroup, we can identify $U_1: SU(2) \to GL(3,\mathbb{C})$.

Excersice 26 :

$$\left.\begin{aligned} \rho(g)\rho(h)\rho(k) &= e^{i\theta(g,h)}\rho(gh)\rho(k) = e^{i\theta(g,h)}e^{i\theta(gh,k)}\rho(ghk) \\ \rho(g)\rho(h)\rho(k) &= e^{i\theta(h,k)}\rho(g)\rho(hk) = e^{i\theta(h,k)}e^{i\theta(g,hk)}\rho(ghk) \end{aligned}\right\} \quad \begin{aligned} &e^{i\theta(g,h)}e^{i\theta(gh,k)} \\ &= \\ &e^{i\theta(h,k)}e^{i\theta(g,hk)}. \end{aligned}$$

Excersice 27 :

If the cocycle were inessential, one could make a choise such that $\theta'(g,h) = 0 \ \forall g,h$, so we would have $V_j(hh') = V_j(h)V_j(h')$.

But this would imply $U_j(\pm gg') = U_j(g)U_j(g') \Rightarrow U_j(-1) = 1$, which is not true for half-integer!

Excersice 28 :

$$x^\mu \sigma_\mu = x_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

$$\Rightarrow \det(x^\mu \sigma_\mu) = (x_0 + x_3)(x_0 - x_3) - (x_1 + ix_2)(x_1 - ix_2) = x_0^2 - x_3^2 - (x_1^2 + x_2^2) = - x^\mu x_\mu .$$

Excersice 29 :

The previous excersice impleies that $\det(T) = - T^\mu T_\mu$ of the vector $(T^0, T^1, T^2, T^3) \in \mathbb{R}^4$.

Since $\det(\rho(g) T) = \det(g T g^*) = \underbrace{\det(g)}_{=1} \cdot \det(T) \cdot \underbrace{\det(g^*)}_{=1} = \det(T)$

it follows that ρ preserves the Minkowski metric.

Hence $\rho : SL(2, \mathbb{C}) \rightarrow O(3,1)$.

Excersice 30 :

ρ maps the identity to the identity of $O(3,1)$, which consits itself of connected components but the one with the identity is $SO_0(3,1)$. And because $SL(2,\mathbb{C})$ consits of only one connected component and ρ is continous, its range lies in $SO_0(3,1)$.

Excersice 31 :

We always have $\rho(g) = \rho(-g)$: $\rho(-g) T = (-g) T (-g)^* = g T g^* = \rho(g) T$.

ρ is exactly two-to-one, because suppose $\rho(g) = \rho(h)$, then $\rho(g h^{-1}) = \rho(g) \rho(h)^{-1} = 1$. The only way we can have $\rho(g h^{-1}) = 1$ is if $g h^{-1}$ commutes with all 2×2 hermitian complex matrices. From excersice 24 it follows that this can only happen if $g h^{-1}$ is a scalar multiple of the identity. The only scalar multiples of the identity that lie in $SL(2,\mathbb{C})$ are ± 1, so we must have $h = \pm g$.

Thus ρ is two-to-one.

Excersice 32 :

Excersice 33 :

It's just the exponential sum and since $\sum_{k=0}^{\infty} \left\| \frac{T^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|T\|^k}{k!} = e^{\|T\|}$ because $\|T\| < \infty$ this sum converges.

Excersice 34 :

First note that $n^x J_x + n^y J_y + n^z J_z = \begin{pmatrix} 0 & -n^z & n^y \\ n^z & 0 & -n^x \\ -n^y & n^x & 0 \end{pmatrix} =: N$

$$(n^x J_x + n^y J_y + n^z J_z)^2 = \begin{pmatrix} -(n^z)^2-(n^y)^2 & n^x n^y & n^z n^x \\ n^x n^y & -(n^z)^2-(n^x)^2 & n^z n^y \\ n^z n^x & n^z n^y & -(n^x)^2-(n^y)^2 \end{pmatrix} = \overbrace{\begin{pmatrix} (n^x)^2 & n^x n^y & n^z n^x \\ n^x n^y & (n^y)^2 & n^z n^y \\ n^z n^x & n^z n^y & (n^z)^2 \end{pmatrix}}^{=: N \otimes N} - \mathbb{1}$$

$$(n^x J_x + n^y J_y + n^z J_z)^3 = \begin{pmatrix} 0 & n^z & -n^y \\ -n^z & 0 & n^x \\ n^y & -n^x & 0 \end{pmatrix} = -N$$

$$(n^x J_x + n^y J_y + n^z J_z)^4 = \begin{pmatrix} (n^z)^2+(n^y)^2 & -n^x n^y & -n^z n^x \\ -n^x n^y & (n^z)^2+(n^x)^2 & -n^z n^y \\ -n^z n^x & -n^z n^y & (n^x)^2+(n^y)^2 \end{pmatrix} = -\left[\begin{pmatrix} (n^x)^2 & n^x n^y & n^z n^x \\ n^x n^y & (n^y)^2 & n^z n^y \\ n^z n^x & n^z n^y & (n^z)^2 \end{pmatrix} - \mathbb{1} \right]$$

So $\exp(t \cdot (n^x J_x + n^y J_y + n^z J_z)) = \mathbb{1} + tN + \frac{t^2}{2!} N^2 + \frac{t^3}{3!} N^3 + \frac{t^4}{4!} N^4 + \ldots$

$$= \mathbb{1} + tN + \frac{t^2}{2!}(N \otimes N - \mathbb{1}) - \frac{t^3}{3!} N - \frac{t^4}{4!}(N \otimes N - \mathbb{1}) + \ldots$$

$$= \mathbb{1} + \left(t - \frac{t^3}{3!} + \ldots\right) N + \left(\frac{t^2}{2!} - \frac{t^4}{4!} + \ldots\right)(N \otimes N - \mathbb{1})$$

$$= \mathbb{1} + \sin t \, N + (1 - \cos t)(N \otimes N - \mathbb{1})$$

$$= \cos t \, \mathbb{1} + \sin t \, N + (1 - \cos t) \, N \otimes N$$

which is the rotation around a unit vector $(n^x, n^y, n^z) \in \mathbb{R}^3$.

Excersice 35 :

$$\exp(sJ_x)\exp(tJ_y) - \exp(tJ_y)\exp(sJ_x) = (\mathbb{1} + sJ_x + \ldots)(\mathbb{1} + tJ_y + \ldots) - (\mathbb{1} + tJ_y + \ldots)(\mathbb{1} + sJ_x + \ldots)$$

$$= \cancel{\mathbb{1}} + \cancel{tJ_y} + \cancel{sJ_x} + stJ_xJ_y + \ldots - (\cancel{\mathbb{1}} + \cancel{tJ_y} + \cancel{sJ_x} + tsJ_yJ_x)$$

$$= st\,J_xJ_y - st\,J_yJ_x + \ldots$$

$$= st\,(J_xJ_y - J_yJ_x) + \ldots$$

Excersice 36 :

$$J_x^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

$$J_y^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$J_z^2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

$$[J_x, J_y] = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = J_z$$

$$[J_y, J_z] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = J_x$$

$$[J_z, J_x] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = J_y .$$

Excersice 37 :

$$\exp((s+t)T) = \sum_{k=0}^{\infty} \frac{(s+t)^k T^k}{k!} = \sum_{k=0}^{\infty}\sum_{i=0}^{k} \binom{k}{i} s^i t^{k-i} \frac{T^k}{k!} = \sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{\cancel{k!}}{i!(k-i)!} s^i t^{k-i} \frac{T^k}{\cancel{k!}}$$

$$= \sum_{k=0}^{\infty}\sum_{i=0}^{k} \left(\frac{s^i T^i}{i!}\right)\cdot\left(\frac{t^{k-i}T^{k-i}}{(k-i)!}\right) = \left(\sum_{k=0}^{\infty}\frac{s^k T^k}{k!}\right)\cdot\left(\sum_{k=0}^{\infty}\frac{t^k T^k}{k!}\right)$$

$$= \exp(sT)\cdot\exp(tT).$$

For a fixed T, $\exp(tT) = \mathbb{1} + tT + \frac{t^2}{2!}T^2 + \dots$ is a polynomial in t, thus smooth.

When $t=0$, then we see from the power series above, $\exp(tT)\big|_{t=0} = \mathbb{1}$.

Moreover $\frac{d}{dt}\exp(tT)\big|_{t=0} = \sum_{n=0}^{\infty} \frac{d}{dt} t^n T^n / n! \big|_{t=0} = T\cdot\sum_{n=1}^{\infty} t^{n-1}T^{n-1}/(n-1)! \big|_{t=0}$

$$= T\cdot\exp(tT)\big|_{t=0} = T\cdot\mathbb{1} = T.$$

Excersice 38 :

Let γ be a path in $gL(n,\mathbb{C})$ with $\gamma(0) = \mathbb{1}$.

Then for every $t\in\mathbb{R}$ we have $\det(\gamma(t)) \neq 0$. Let $\gamma(t) = \exp(tT)$ for T any complex matrix.

Then with the next excersice we find that $\det(\exp(tT)) = e^{tr(T\cdot t)} \neq 0$.

So the Lie algebra of $gL(n,\mathbb{C})$ consists of all $n\times n$ complex matrices.

Same for $gL(n,\mathbb{R})$.

Excersice 39 :

Let T be a diagonalizable matrix $\Rightarrow \exists S$, so that $D = S^{-1}TS$ is diagonal.

$$\Rightarrow \det(\exp(T)) = \det(\exp(SDS^{-1})) = \det\left(\mathbb{1} + SDS^{-1} + \frac{(SDS^{-1})^2}{2!} + \dots\right)$$

$$= \det\left(\mathbb{1} + SDS^{-1} + \frac{SD^2S^{-1}}{2!} + \dots\right) = \det\left(\sum_{k=0}^{\infty} S\frac{D^k}{k!}S^{-1}\right)$$

$$= \det(S\exp(D)S^{-1}) = \det(\exp(D))$$

$$= \det\begin{pmatrix} e^{d_1} & & \\ & \ddots & \\ & & e^{d_n}\end{pmatrix} = \prod_{i=0}^{n} e^{d_i} = e^{\sum_{i=0}^{n} d_i} = e^{tr(D)} = e^{tr(T)}$$

Since $tr(D) = tr(S^{-1}TS) = tr(S^{-1}ST) = tr(T)$.

Since the diagonalizable matrices are dense in the space of all matrices, we can reach every matrix as a sequence of diagonalizable ones. So this resuel is valid for every matrix T.

Let $\gamma(t) = \exp(tT)$ with $T\in\{n\times n$ traceless real/complex matrices$\}$.

Then $\det(\gamma(t)) = e^{tr(t\cdot T)} = e^{t\cdot tr(T)} = e^0 = 1 \Rightarrow \gamma(t)\in SL(n,\mathbb{R}/\mathbb{C})$.

So the Lie algebras of $SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$ consist of all traceless matrices.

Excersice 40 :

Let $\gamma(t)$ be a path in $SO(p,q) \Rightarrow g(\gamma(t)v, \gamma(t)w) = g(v,w)$

$\Rightarrow g(\gamma'(t)v, \gamma(t)w) + g(\gamma(t)v, \gamma'(t)w) = 0 \Longrightarrow g(Tv,w) = -g(v,Tw)$

Thus these matrices have to satisfy $T_{ij} = -T_{ji}$. The dimension of all skew-adjoint $n \times n$-matrices is $n(n-1)/2$. So $\dim(\mathfrak{so}(p,q)) = \dim(SO(p,q)) = \frac{n(n-1)}{2}$.

To determine an explicit basis of $\mathfrak{so}(3,1)$, we have to find 6 elements.

Let $\gamma(\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ be a path in $SO(3,1)$.

Then $\gamma'(0) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. If we do the same for the other t-x^i-mixings and the pure space mixing, we get:

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & J_x & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & J_y & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & J_z & \\ 0 & & & \end{pmatrix}.$$

Excersice 41:

The Lie algebra of $U(n)$ consits of all skew-adjoint complex $n \times n$ matrices, because for every path $\gamma(t)$ in $U(n)$, they have to satisfy $\langle \gamma(t)v, \gamma(t)w \rangle = \langle v, w \rangle$

$\Rightarrow \langle \gamma'(0)v, w \rangle + \langle v, \gamma'(0)w \rangle = 0$

$\Leftrightarrow \sum_{i=1}^{n} \overline{T}_{ij} \bar{v}_j w_i = \sum_{i=1}^{n} -\bar{v}_i T_{ik} w_k$

$\Leftrightarrow T_{ij} = -\overline{T}_{ji}$.

For $U(1)$ we get therefore: $\mathfrak{u}(1) = \{ z \in \mathbb{C} : z = -\bar{z} \} = \{ ix : x \in \mathbb{R} \}$.

For the Lie algebra of $SU(n)$, we just take all elements of $U(n)$ with determinant 1.

We have seen that $\exp(tT) \in U(n)$, if T is skew-adjoint complex matrix.

Moreover $\det(\exp(tT)) = 1$ if $\operatorname{tr}(T) = 0$, so $\mathfrak{su}(n) = \{$traceless skew-adjoint complex $n \times n$ mat$\}$.

Excersice 42:

$\gamma(t)\gamma^{-1}(t) = 1 \quad \Rightarrow \quad 0 = \frac{d}{dt}\left(\gamma(t)\gamma^{-1}(t)\right) = \gamma'(t)\gamma^{-1}(t) + \gamma(t)(-1)\frac{1}{\gamma^2(t)}\gamma'(t)$

$\Rightarrow \frac{d}{dt}\gamma^{-1}(t) = -\frac{1}{\gamma^2(t)}\frac{d}{dt}\gamma(t) \overset{\gamma(0)=1}{\Longrightarrow} \frac{d}{dt}\gamma^{-1}(t)\Big|_{t=0} = -\frac{d}{dt}\gamma(t)\Big|_{t=0}$.

Excersice 43:

$\frac{d}{dt}\gamma(t)\eta(t)\Big|_{t=0} = \frac{d}{dt}\gamma(t)\Big|_{t=0}\eta(0) + \gamma(0)\frac{d}{dt}\eta(t)\Big|_{t=0} = \frac{d}{dt}\gamma(t)\Big|_{t=0} + \frac{d}{dt}\eta(t)\Big|_{t=0}$.

So the differential of $\cdot : g \times g \to g$ at $(1,1) \in g \times g$ is the addition map $\mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$

$\left(\frac{d}{dt}\gamma(t)\Big|_{t=0}, \frac{d}{dt}\eta(t)\Big|_{t=0}\right) \mapsto \gamma'(0) + \eta'(0)$

Excersice 44:

1) $[v,w] = vw - wv = -(wv - vw) = -[w,v]$.

2) $[u, \alpha v + \beta w] = u(\alpha v + \beta w) - (\alpha v + \beta w)u = \alpha uv + \beta uw - \alpha vu - \beta wu = \alpha[u,v] + \beta[u,w]$.

3) $[u,[v,w]] + [v,[w,u]] + [w,[u,v]]$

$= u(vw-wv) - (vw-wv)u + v(wu-uw) - (wu-uw)v + w(uv-vu) - (uv-vu)w.$

$= uvw - \cancel{uwv} - \cancel{vwu} + \cancel{wvu} + \cancel{vwu} - \cancel{vuw} - \cancel{wuv} + \cancel{uwv} + \cancel{wuv} - \cancel{wvu} - \cancel{uvw} + \cancel{vuw}$

$= 0.$

Excersice 45 :

We have shown that the Lie algebra $u(n)$ of $U(n)$ consists of all skew-adjoint complex $n\times n$-matrices and that $su(n)$ consists of all traceless skew-adjoint complex $n\times n$-matrices. For $SU(2)$, we have seen that it is isomorphic to the 3-sphere S^3, so the tangent space or the Lie algebra will be 3-dimensional.

Now
$$I = -i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad J = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad K = -i\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$
are three linear independent traceless skew-adjoint complex 2×2-matrices, so they build a basis for $su(2)$.

The linear map $f: su(2) \to so(3)$ given by $-\frac{i}{2}\sigma_j \mapsto J_j$ is a Lie algebra isomorphism:

- We have seen that $\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k$, so $[\sigma_i,\sigma_j] = 2i\varepsilon_{ijk}\sigma_k$.

$$\Rightarrow \quad f([-\tfrac{i}{2}\sigma_i, -\tfrac{i}{2}\sigma_j]) = -\tfrac{1}{4} f([\sigma_i,\sigma_j]) = -\tfrac{1}{4} f(2i\varepsilon_{ijk}\sigma_k) = -\tfrac{1}{2} i\varepsilon_{ijk} \overbrace{f(\sigma_k)}^{=2iJ_k} = \quad \varepsilon_{ijk}J_k.$$
$$[f(-\tfrac{i}{2}\sigma_i), f(-\tfrac{i}{2}\sigma_j)] = [J_i, J_j] = \varepsilon_{ijk}J_k. \quad ✓$$

- Since f is one-to-one and both Lie algebras are 3-dimensional, f is a isomorphism.

Excersice 46 :

Let ϕ be a diffeomorphism of M and $f \in \mathcal{F}(M)$:

$$\phi_*(v)(f)(\phi(p)) = v(f\circ\phi)(p)$$
$$\phi_*[v,w](f) = [v,w](\phi^* f) = v(w(\phi^* f)) - w(v(\phi^* f)) \overset{\downarrow}{=} v(\phi_* w(f)\circ\phi) - w(\phi_* v(f)\circ\phi).$$
$$[\phi_* v, \phi_* w](f) = \phi_* v(\phi_* w(f)) - \phi_* w(\phi_* v(f)) = \phi_* v(w(\phi^* f)) - \phi_* w(v(\phi^* f))$$
$$= v(w(\phi^* f)\circ\phi) - w(v(\phi^* f)\circ\phi) = v(\phi_* w(f)\circ\phi) - w(\phi_* v(f)\circ\phi).$$
$$\Rightarrow \quad \phi_*[v,w] = [\phi_* v, \phi_* w].$$

If v, w are two left-invariant vector fields then $\phi_*[v,w] = [\phi_* v, \phi_* w] = [v,w]$ and so the Lie bracket is also left-invariant.

Excersice 47 :

With $\phi_t(g) = g\exp(tv_1)$ we have $\frac{d}{dt}\phi_t(g)\big|_{t=0} = g v_1 \exp(0) = (L_g)_* v_1 = v_g.$

Excersice 48 :

Let u_1, v_1 and $w_1 = [u_1, v_1]$ be matrices in $\mathfrak{g}$ and u, v, w the corresponding left-invariant vector fields on g. We can write $\phi_t(g) = g\exp(tv_1)$, $\Psi_t(g) = g\exp(tu_1)$, $\chi_t(g) = g\exp(tw_1)$.

$$\Rightarrow \quad [u,v] = \left[\tfrac{d}{dt}\Psi_t(g)\big|_{t=0}, \tfrac{d}{dt}\phi_t(g)\big|_{t=0}\right] = [(L_g)_* u_1, (L_g)_* v_1] = (L_g)_*[u_1, v_1] = (L_g)_* w_1$$
$$= \tfrac{d}{dt}\chi_t(g)\big|_{t=0} = w. \quad ✓$$

$\Rightarrow \quad \mathfrak{g} \cong$ left-invariant vector fields on g

Excersice 49 :

Since $d\varrho = (\varrho)_* : T_1 G \rightarrow T_1 H$ just amounts to pushing forward tangent vectors, we can just use the property of the pushforward, $(\varrho)_* [v, w] = [(\varrho)_* v, (\varrho)_* w]$.
Thus $d\varrho$ is a Lie algebra homomorphism.

Excersice 50 :

$$\varrho(g_t)\sigma_2 = \begin{pmatrix} e^{-it/2} & 0 \\ 0 & e^{it/2} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix} = \begin{pmatrix} e^{-it/2} & 0 \\ 0 & e^{it/2} \end{pmatrix} \begin{pmatrix} 0 & -ie^{-it/2} \\ ie^{it/2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -ie^{-it} \\ ie^{it} & 0 \end{pmatrix}$$

$$\varrho(g_t)\sigma_3 = \begin{pmatrix} e^{-it/2} & 0 \\ 0 & e^{it/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$$

$$-\sin t\,\sigma_1 + \cos t\,\sigma_2 = \begin{pmatrix} 0 & -\frac{1}{2i}(e^{it} - e^{-it}) \\ -\frac{1}{2i}(e^{it} - e^{-it}) & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{i}{2}(e^{it} + e^{-it}) \\ \frac{i}{2}(e^{it} + e^{-it}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -ie^{-it} \\ ie^{it} & 0 \end{pmatrix} = \varrho(g_t)\sigma_2 \quad \checkmark$$

Excersice 51 :

Do essentially the same steps as in the book :

Write $g_t = \exp(-it\sigma_1/2)$, calculate it and find $\varrho(g_t)$ by evaluating $\varrho(g_t)\sigma_j = g_t \sigma_j g_t^{-1}$.

Note $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow g_t = \mathbb{1} + \begin{pmatrix} 0 & -it/2 \\ -it/2 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -t^2/4 & 0 \\ 0 & -t^2/4 \end{pmatrix} + \frac{1}{3!} \begin{pmatrix} 0 & it^3/8 \\ it^3/8 & 0 \end{pmatrix} + \frac{1}{4!} \begin{pmatrix} t^4/16 & 0 \\ 0 & t^4/16 \end{pmatrix} + \dots$

$$= \mathbb{1} - i\sin(t/2)\sigma_1 + (\cos(t/2)\cdot\mathbb{1} - \mathbb{1})$$

$$= \cos(t/2)\cdot\mathbb{1} - i\sin(t/2)\sigma_1$$

$$\Rightarrow \quad g_t = \begin{pmatrix} \cos(t/2) & -i\sin(t/2) \\ -i\sin(t/2) & \cos(t/2) \end{pmatrix}$$

$$g_t^{-1} = \begin{pmatrix} \cos(t/2) & i\sin(t/2) \\ i\sin(t/2) & \cos(t/2) \end{pmatrix}$$

- $\varrho(g_t)\sigma_1 = \begin{pmatrix} \cos(t/2) & -i\sin(t/2) \\ -i\sin(t/2) & \cos(t/2) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(t/2) & i\sin(t/2) \\ i\sin(t/2) & \cos(t/2) \end{pmatrix}$

$$= \begin{pmatrix} \cos(t/2) & -i\sin(t/2) \\ -i\sin(t/2) & \cos(t/2) \end{pmatrix} \begin{pmatrix} i\sin(t/2) & \cos(t/2) \\ \cos(t/2) & i\sin(t/2) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$$

- $\varrho(g_t)\sigma_2 = \begin{pmatrix} \cos(t/2) & -i\sin(t/2) \\ -i\sin(t/2) & \cos(t/2) \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos(t/2) & i\sin(t/2) \\ i\sin(t/2) & \cos(t/2) \end{pmatrix} = \begin{pmatrix} \cos(t/2) & -i\sin(t/2) \\ -i\sin(t/2) & \cos(t/2) \end{pmatrix} \begin{pmatrix} \sin(t/2) & -i\cos(t/2) \\ i\cos(t/2) & -\sin(t/2) \end{pmatrix}$

$$= \begin{pmatrix} 2\sin(t/2)\cos(t/2) & i\sin^2(t/2) - i\cos^2(t/2) \\ i\cos^2(t/2) - i\sin^2(t/2) & -2\sin(t/2)\cos(t/2) \end{pmatrix} = \begin{pmatrix} \sin(t) & -i\cos(t) \\ i\cos(t) & -\sin(t) \end{pmatrix} = \sin t\,\sigma_3 + \cos t\,\sigma_2$$

- $\varrho(g_t)\sigma_3 = \begin{pmatrix} \cos(t/2) & -i\sin(t/2) \\ -i\sin(t/2) & \cos(t/2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(t/2) & i\sin(t/2) \\ i\sin(t/2) & \cos(t/2) \end{pmatrix} = \begin{pmatrix} \cos(t/2) & -i\sin(t/2) \\ -i\sin(t/2) & \cos(t/2) \end{pmatrix} \begin{pmatrix} \cos(t/2) & i\sin(t/2) \\ -i\sin(t/2) & -\cos(t/2) \end{pmatrix}$

$$= \begin{pmatrix} \cos^2(t/2) - \sin^2(t/2) & 2i\sin(t/2)\cos(t/2) \\ -2i\sin(t/2)\cos(t/2) & \sin^2(t/2) - \cos^2(t/2) \end{pmatrix} = \begin{pmatrix} \cos(t) & i\sin(t) \\ -i\sin(t) & -\cos(t) \end{pmatrix} = \cos t\,\sigma_3 - \sin t\,\sigma_2$$

$\Rightarrow \quad \varrho(g_t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}$ which is a rotation around the x-axis as desired.

Excersice 52 :

The spin-1/2 representation of $SU(2)$ corresponds to the fundamental matrix representation.

$$\left.\begin{aligned} &\cdot\ \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1/2\\0\end{pmatrix}\right\rangle = \frac{1}{2} \\ &\left\langle \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\-1/2\end{pmatrix}\right\rangle = -\frac{1}{2} \end{aligned}\right\} \text{ about the z-axis } (\sigma_3)$$

$$\left.\begin{aligned} &\cdot\ \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\\frac{1}{2}i\end{pmatrix}\right\rangle = 0 \\ &\left\langle \begin{pmatrix}0\\1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-\frac{1}{2}i\\0\end{pmatrix}\right\rangle = 0 \end{aligned}\right\} \text{ about the y-axis } (\sigma_2)$$

$$\left.\begin{aligned} &\cdot\ \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1/2\end{pmatrix}\right\rangle = 0 \\ &\left\langle \begin{pmatrix}0\\1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1/2\\0\end{pmatrix}\right\rangle = 0 \end{aligned}\right\} \text{ about the x-axis } (\sigma_1)$$

Excersice 53 :

- $\mathfrak{sl}(n, \mathbb{R}/\mathbb{C})$ consists of all traceless real/complex $n\times n$-matrices.

  Let $A, B \in \mathfrak{sl}(n, \mathbb{R}/\mathbb{C})$, then $[A,B] = AB - BA \in \mathfrak{sl}(n, \mathbb{R}/\mathbb{C})$ because

  $\mathrm{tr}([A,B]) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = \mathrm{tr}(AB) - \mathrm{tr}(AB) = 0$.

  $\forall C \in \mathfrak{sl}(n, \mathbb{R}/\mathbb{C})$ you can find such A, B that C is a linear combination of Lie brackets.

  In the 1-dimensional case we have just $\mathfrak{sl}(1, \mathbb{R}, \mathbb{C}) = \{0\}$.

- $\mathfrak{so}(p,q)$ consists of all $n\times n$ real matrices with $g(Tv, w) = -g(v, Tw)$.

  Let $A, B \in \mathfrak{so}(p,q) \Rightarrow g([A,B]v, w) = g(ABv - BAv, w) = g(ABv, w) - g(BAv, w)$

  $\Rightarrow [A,B] \in \mathfrak{so}(p,q)$. $\quad = g(v, BAw) - g(v, ABw) = -g(v, [A,B]w)$

- $\mathfrak{su}(n)$ consists of all traceless skew-adjoint complex $n\times n$-matrices.

  Let $A, B \in \mathfrak{su}(n) \Rightarrow [A,B]_{ij} = A_{ik}B_{kj} - B_{ik}A_{kj} = \overline{B_{jk}A_{ki}} - \overline{A_{jk}B_{ki}} = -\overline{[A,B]_{ji}}$

  $\Rightarrow [A,B] \in \mathfrak{su}(n)$ because also traceless.

  $\mathfrak{su}(1) = \{0\}$.

Excersice 54 :

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, so is the direct sum $\mathfrak{g}\oplus\mathfrak{h}$ because conditions 1-3 holds and it is closed:

$$[\underset{\mathfrak{g}\oplus\mathfrak{h}}{\underset{\wedge}{(x,x')}}, \underset{\mathfrak{g}\oplus\mathfrak{h}}{\underset{\wedge}{(y,y')}}] = (\underset{\mathfrak{g}}{\underset{\wedge}{[x,y]}}, \underset{\mathfrak{h}}{\underset{\wedge}{[x',y']}}) \in \mathfrak{g}\oplus\mathfrak{h}.$$

If $\mathfrak{g}$ and $\mathfrak{h}$ are semisimple, we can write every element $A \in \mathfrak{g}$ $A = [x,y]$ and $B \in \mathfrak{h}$ $B = [x',y']$

So $(A,B) = ([x,y], [x',y']) = [(x,x'), (y,y')]$, so $\mathfrak{g}\oplus\mathfrak{h}$ is also semisimple.

Excersice 55 :

If we define $V_\alpha = \{ v \in TM : \pi(v) \in U_\alpha \}$, where $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ are charts for M, then every point in TM (p, T_pM) lies in some set V_α because $\pi : TM \to M$ is onto and so for every $v \in TM = \bigcup_{p \in M} T_pM$ there is a tangent space T_pM such that $\pi(v) = p \in U_\alpha$.
Further, note that since φ_α are smooth, so is $\psi_\alpha : V_\alpha \to \mathbb{R}^n \times \mathbb{R}^n$, $\psi_\alpha(v) = (\varphi_\alpha(\pi(v)), (\varphi_\alpha)_* v)$.
Transitionfunctions are smooth likewise, since they are compositions of smooth ψ_α and ψ_β^{-1}.
The projection π is smooth because we can define a $\mathbb{R}^{2n} \to \mathbb{R}^n$ version of the map by ignoring the last n coordinates (which is smooth) but that is just $\varphi_\alpha \circ \pi \circ \psi_\alpha^{-1}$, so π must be smooth.

Excersice 56 :

$\psi : E \to E'$, $\phi : M \to M'$ are a bundle morphism $\Leftrightarrow$ $\pi' \circ \psi = \phi \circ \pi$.

„$\Rightarrow$" ψ maps each fibes E_p into the fibes $E'_{\phi(p)}$.

So
$$\left.\begin{aligned} (\pi' \circ \psi)(p, v) &= \pi'(\phi(p), v') = \phi(p) \\ (\phi \circ \pi)(p, v) &= \phi(\pi(p, v)) = \phi(p) \end{aligned}\right\} \Rightarrow \pi' \circ \psi = \phi \circ \pi$$

„$\Leftarrow$" If the equation is fullfilled then ψ must map the fibes E_p into $E'_{\phi(p)}$ (see above).

Excersice 57 :

We can see that $\psi'_\alpha \circ \varphi_* \circ \psi_\beta^{-1}$ is smooth so have φ_* to be.
Or since the right side of $\pi' \circ \phi_* = \phi \circ \pi$ is smooth, ϕ_* has to be smooth.

Excersice 58 :

When ϕ is an isomorphism, the dimensions of M and M' are the same and the spaces T_pM and $T_{\phi(p)}M'$ are isomophic. Since ϕ_* is smooth and linear, it is an isomorphism of the tangenspaces.

Excersice 59 :

The induced charts $\psi_\alpha : V_\alpha \to \mathbb{R}^n \times \mathbb{R}^n$ give us a local trivialisation with $\mathbb{R}^n$ as standard fiber.

Excersice 60 :

Excersice 61 :

The standard fiber of the tangent bundle is $\mathbb{R}^n$ and the trivialisation is linear because it's just the pushforward.

Excersice 62 :

Locally the Möbius strip just look like $S^1 \times \mathbb{R}$.

Excersice 63 :

Excersice 64 :

A section of the tangent bundle assigns to each point in the base space a vector in its tangent space. From this we obtain a vector field from the pointwise action of the tangent vectors. The output of the vector field is again smooth since the section is smooth. The directional derivative changes smoothly from point to point since the tangent vectors do.

Excersice 65 :

The first conditions are defined in the text. The other two are simply

$$\big((fg)s\big)(p) = (fg)(p)\, s(p) = f(p)\, g(p)\, s(p) = \big(f(gs)\big)(p) \qquad \forall f,g \in C^\infty(M).$$

Excersice 66 :

There is one segment where the Möbius strip is twisted, so we have $(0,v) \triangleq (2\pi,-v)$.
If we have a section s, it has to satisfy $s(0) = v = -s(2\pi) = -s(0) \Rightarrow s(0) = 0$.
So every section vanishes somewhere $\Rightarrow$ the sections are not linearly independent
$\Rightarrow$ we have no basis of sections
$\Rightarrow$ the Möbius strip is not trivial.

Excersice 67 :

The basic idea, is that since $E|_U$ is locally equivalent to $U \times \mathbb{R}^n$ for any open set U, then since E^* just replaces E_p with E_p^* everywhere, $E^*|_U$ must be locally equivalent to $U \times (\mathbb{R}^n)^* = U \times \mathbb{R}^n$. To see that it is a vector bundle, just use the canonical dual basis when mapping E_p^* to $\mathbb{R}^n$.

Excersice 68 :

Locally, the vector bundles are trivial, so the sections look like a funktion from M to V, where $E_p = \{p\} \times V$. The function is smooth since the sections are smooth.
$\lambda(s)$ is linear over λ and s since the action of the cotangent bundle on the tangent bundle is linear in each argument.
We get the $C^\infty(M)$ part since the action is pointwise and smooth.

Excersice 69 :

A section of the cotangent bundle asigns to each point $p \in M$ a cotangent vector, which is a linear function from T_pM to $\mathbb{R}$. A 1-form is a map taking vector fields to functions on the manifold. We see that the sections acts pointwise as $s(p) : T_pM \to \mathbb{R}$, so $s : TM \to C^\infty(M)$, which is a 1-form.

Excersice 70 :

If E and E' are vector bundles over M, there exist a local trivialization $\phi : E|_U \to U \times \mathbb{R}^n$ and $\phi' : E'|_{U'} \to U' \times \mathbb{R}^m$. Define
$$\psi : E \oplus E'|_{U \cap U'} \to (U \cap U') \times \mathbb{R}^{n+m}.$$
$$\theta : E \otimes E'|_{U \cap U'} \to (U \cap U') \times \mathbb{R}^{n \cdot m}$$

Excersice 71 :

clear

Excersice 72 :

Every section of $E \otimes E'$ can be uniquely written as $(s \otimes s')(p) = s(p) \otimes s'(p)$ where $s \in \Gamma(E)$, $s' \in \Gamma(E')$.
Suppose $s = \sum_i \sigma_i s^i$ and $s' = \sigma'_i s'^i$ $\Rightarrow$ $(s \times s')(p) = \sum_{i,j} \sigma_i \sigma'_j \, s^i(p) \otimes s'^j(p)$

If E, E' don't have a basis of sections this expressions could be not unique.

Excersice 73 :

Just use the fact that E is a vector bundle and use their charts $\varphi_\alpha : E \to \mathbb{R}^n$ to get
$$\phi_\alpha : \Lambda E \to \Lambda \mathbb{R}^n , \quad (p, v_1 \wedge \dots \wedge v_p) \mapsto (\varphi_\alpha(p), \varphi_\alpha(v_1) \wedge \dots \wedge \varphi_\alpha(v_p)),$$
which are charts for ΛE (making ΛE into a manifold) and local trivialisations at the same time.

Excersice 74 :

We know that $\Lambda \mathbb{R}^n = \bigoplus_{i=0}^{n} \Lambda^i \mathbb{R}^n$, so if we use charts we can pullback this property on our manifold so that $\Lambda E = \bigoplus_{i=0}^{n} \Lambda^i E$.

Since $\Lambda^0 E \cong \mathbb{R}$, sections of this are just functions from M to $\mathbb{R}$.
$\Lambda^1 E \cong E$, -||- -||- M to E $\Rightarrow$ sections of E.

Excersice 75 :

With the definition $(\omega \wedge \mu)(p) = \omega(p) \wedge \mu(p)$, sections of ΛE form an algebra in a natural way, which one can see if we pullback to $\mathbb{R}^n$.

- $\Lambda^i E$ form a subspace of ΛE
- sections of $\Lambda^i E$ are locally finite sums of wedge products of sections of E } properties like in VS $\omega_1, \dots, \omega_i \in \Gamma(E) \Rightarrow \omega_1 \wedge \dots \wedge \omega_i \in \Gamma(\Lambda^i E)$

Excersice 76 :

T^*M is the (vector space) of 1-forms. We can build sections of $\Lambda^i T^*M$ as finite sums of sections of T^*M (1-forms) of the form $dx_1 \wedge \dots \wedge dx_i$, which are in fact i-forms.
$$(s_1 \wedge \dots \wedge s_i)(p) = s_1(p) \wedge \dots \wedge s_i(p) = \prod_{k=1}^{i} a_k \, dx_1 \wedge \dots \wedge dx_i , \quad \text{where } s_k(p) = a_k dx_k .$$

Excersice 77

If $p \in U_\alpha \cap U_\beta$ we identify $(p, v) \in U_\alpha \times V$ with $(p, g_{\alpha\beta} g_{\beta\alpha} v) \in U_\alpha \times V$.

$$\Rightarrow \quad g_{\alpha\beta} g_{\beta\alpha} = 1 \quad \text{on } U_\alpha \cap U_\beta \quad \Rightarrow \quad g_{\alpha\beta} = g_{\beta\alpha}^{-1} .$$

If we have any sequences $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_m$ with $\alpha_1 = \beta_1$ and $\alpha_n = \beta_m$, so the same start and end point, they have to lead to the same result!

$$v = g_{\alpha_1\alpha_2} \cdots g_{\alpha_{n-1}\alpha_n} v' = g_{\underset{\overset{\|}{\alpha_1}}{\beta_1}\beta_2} \cdots g_{\beta_{m-1}\underset{\overset{\|}{\alpha_n}}{\beta_m}} v' .$$

Excersice 78:

Set $E_p = \pi^{-1}(\{p\})$, if $p \in U_\alpha \Rightarrow \phi_\alpha|_p : E_p \to \{p\} \times V$ is bijective.
So the structure of the vector space comes from this isomorphism, and the conditions garantees that everything is fine. For each set U_α, there is a fiberwise linear local trivialization and the cocycle condition garantees that identified points in E are mapped into the same points in $U_\alpha \times V$. For example $\phi_\alpha[p, v]_\alpha = (p, v)$ and $\underbrace{(\phi_\alpha \circ \phi_\beta^{-1})}_{= g_{\alpha\beta}} \underbrace{(\phi_\beta[p, g_{\beta\alpha} v]_\beta)}_{= (p, g_{\beta\alpha} v)} = (p, v)$.

Excersice 79:

$T: E_p \to E_p$ lives in $\mathfrak{g}$ if it is of the form $[p, v]_\alpha \mapsto [p, d\rho(x) v]_\alpha$ for some $x \in \mathfrak{g}$.

$$\left.\begin{array}{l} \text{If } p \in U_\alpha \cap U_\beta \Rightarrow [p, v]_\alpha = [p, g_{\beta\alpha} v]_\beta \\ \text{Similarly } [p, d\rho(x) v]_\alpha = [p, g_{\beta\alpha} d\rho(x) v]_\beta \end{array}\right\} \quad [p, g_{\beta\alpha} v]_\beta \mapsto [p, g_{\beta\alpha} d\rho(x) v]_\beta$$

Or if we define $v' = g_{\beta\alpha} v$ and $d\rho(x') = g_{\beta\alpha} d\rho(x) g_{\alpha\beta}$: $[p, v']_\beta \mapsto [p, d\rho(x') v']_\beta$.

Excersice 80:

$U_1(g)$ is just a rotation and we know that they preserve the inner product, so,

$$\begin{aligned} &(\partial_\mu \partial^\mu + m^2) U_1(g)\phi + \lambda \langle U_1(g)\phi, U_1(g)\phi \rangle U_1(g)\phi \\ = &\ U_1(g) (\partial_\mu \partial^\mu + m^2)\phi + \lambda \langle \phi, \phi \rangle U_1(g)\phi \\ = &\ U_1(g) \underbrace{\left[(\partial_\mu \partial^\mu + m^2)\phi + \lambda \phi^i \phi_i \phi \right]}_{= 0} \end{aligned}$$

$\Rightarrow$ $U_1(g)\phi$ is also a solution!

Excersice 81:

All $C^\infty(M)$-linear maps $T: \Gamma(E) \to \Gamma(E)$ correspond to sections of $\mathrm{End}(E)$.

Excersice 82:

Suppose $f, h \in T$ live in $\mathcal{G}$. This means $[p, v]_\alpha \xmapsto{f} [p, g_f v]_\alpha$ for $g_f \in \mathcal{G}$ and similar for h.
$fh: [p, v]_\alpha \mapsto [p, g_{fh} v]_\alpha$ for $g_{fh} = g_f g_h \in \mathcal{G}$, so it lives in $\mathcal{G}$.
When $h = f^{-1}$, we have trivial action $e = g_{ff^{-1}} = g_f g_{f^{-1}}$, thus $g_{f^{-1}} = g_f^{-1}$.
They are all gauge transformations.

Excersice 83:

$A(v) = \sum_i \omega_i(v) T_i$ is well-defined.

Excersice 84:

If we choose a local trivialization, this sets D^0 as follows. From the LT we get a basis of sections e_j.

$D^0_v(e_j) = 0$ implies $D^0_v s = D^0_v(s^j e_j) = v(s^j) e_j + s^j D^0_v e_j = v(s^j) e_j$.

Let's check that $D^0 + A$ is a connection:

1. $D_v(\alpha s) = D^0_v(\alpha s) + A(v)(\alpha s) = \alpha D^0_v s + \alpha A(v) s = \alpha D_v(s)$
2. $D_v(s+t) = D^0_v(s+t) + A(v)(s+t) = D_v(s) + D_v(t)$
3. $D_v(fs) = D^0_v(fs) + A(v)(fs) = v(f)s + f D^0_v s + f A(v) s = v(f)s + f D_v s$
4. $D_{v+w}(s) = D^0_{v+w}(s) + A(v+w)s = D^0_v s + D^0_w s + A(v)s + A(w)s = D_v s + D_w s$
5. $D_{fv}(s) = D^0_{fv}(s) + A(fv)s = f D^0_v(s) + f A(v) s = f D_v s$

Is $D - D^0$ an End(E)-valued 1-form?

1. It is linear in v, since D and D^0 are.
2. It is linear in s, since $D_v(fs) - D^0_v(fs) = \cancel{v(f)s} + f D_v s - \cancel{v(f)s} - f D^0_v s = f(D_v - D^0_v)s$

We can find a vector potential A for every connection D:

$A = A^j_{\mu i}\, e_j \otimes e^i \otimes dx^\mu$ with $A^j_{\mu i} e_j = (D - D^0)(\partial_\mu) e_i = (D_\mu - D^0_\mu) e_i$.
$= A(\partial_\mu) e_i$

Excersice 85:

$D'_v(s) = g D_v(g^{-1}s)$ is a connection:

1. $D'_v(\alpha s) = g D_v(g^{-1}\alpha s) = g \alpha D_v(g^{-1}s) = \alpha D'_v(s)$
2. $D'_v(s+t) = g D_v(g^{-1}(s+t)) = g D_v(g^{-1}s + g^{-1}t) = g D_v(g^{-1}s) + g D_v(g^{-1}t) = D'_v(s) + D'_v(t)$
3. $D'_v(fs) = g D_v(g^{-1}fs) = g D_v(f g^{-1}s) = g v(f) g^{-1}s + g f D_v(g^{-1}s) = v(f)s + f D'_v(s)$
4. $D'_{v+w}(s) = g D_{v+w}(g^{-1}s) = g D_v(g^{-1}s) + g D_w(g^{-1}s) = D'_v(s) + D'_w(s)$
5. $D'_{fv}(s) = g D_{fv}(g^{-1}s) = g f D_v(g^{-1}s) = f D'_v(s)$.

Excersice 86:

When we write the $\mathcal{G}$-connection as $D = D^0 + A$, we get

$$D'_v(s) = g D_v(g^{-1}s) = g D^0_v(g^{-1}s) + g A(v) g^{-1}s = g\, v(g^{-1}) s + g g^{-1} D^0_v s + g A(v) g^{-1} s$$
$$= v^\mu g \partial_\mu g^{-1} s + D^0_v s + v^\mu g A_\mu g^{-1} s$$
$$= D^0_v s + v^\mu \underbrace{(g A_\mu g^{-1} + g \partial_\mu g^{-1})}_{= A'_\mu} s$$

For $g \in \mathcal{G}$, $g \partial_\mu g^{-1}$ lives in $\mathfrak{g}$ because it really means $\rho(g)^j_l\, \partial_\mu \rho(g)^l_k (x^\nu)$ and $\rho(g)(x^\nu) = e^{-c_j h^j}$.
Thus $g \partial_\mu g^{-1} = [\partial_\mu c_j(x^\nu)]\, g h^j g^{-1}$ and since the inverse of a gauge trf. is a gauge trf. $\Rightarrow$ A'_μ lives in $\mathfrak{g}$.

Excersice 87 :

Any $\mathfrak{g}$-connection on E is gauge-equivalent to one in temporal gauge ($A_0 = A(\partial_t) = 0$).
If we apply a gauge transformation g to the vector potential, we get

$$A'_\mu = g A_\mu g^{-1} + g \partial_\mu g^{-1} .$$

If we write $g = e^{-T_i}$ with the generators T_i, we get

$$A'_\mu = g\,(A_\mu + \partial_\mu T_i)\,g^{-1} .$$

Since the vector potential lives in the same space as the generators, which form a basis, we can elliminate A_0 by choosing such a linear combination T_i of generators with $\partial_t T_i = -A_0$.

$\Rightarrow A'_0 = 0$.

Excersice 88 :

First, we have to know the transformation rule for $A \rightarrow A'$. Since $D_\mu e_j = A^i_{\mu j} e_i$:

$$D_{\mu'} e_{j'} = \frac{\partial x^\mu}{\partial x^{\mu'}} D_\mu\left(\frac{\partial x^j}{\partial x^{j'}} e_j\right) = \frac{\partial x^\mu}{\partial x^{\mu'}}\left[D^0_\mu\left(\frac{\partial x^j}{\partial x^{j'}}\right)e_j + \frac{\partial x^j}{\partial x^{j'}} A^i_{\mu j} e_i\right] = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^j}{\partial x^{j'}} A^i_{\mu j} e_i + \frac{\partial^2 x^j}{\partial x^{\mu'}\partial x^{j'}} e_j$$

$$\Rightarrow \quad A^{i'}_{\mu' j'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{i'}}{\partial x^i} A^i_{\mu j} + \frac{\partial^2 x^j}{\partial x^{\mu'}\partial x^{j'}}\frac{\partial x^{i'}}{\partial x^j} .$$

If we write $D_{\gamma'(t)} u(t) = \frac{d}{dt} u(t) + A(\gamma'(t))\,u(t)$ in local coordinates, we get

$$\frac{d}{dt} v^{i'}(t) + A^{i'}_{\mu' j'} \gamma^{\mu'} v^{j'} = \frac{\partial x^{i'}}{\partial x^i}\left[\frac{d}{dt} v^i(t) + A^i_{\mu j}\gamma^\mu v^j\right]$$

after a change of coordinates (change of local trivialization). This shows that the covariant derivative defined in this manner is a tensor and thus independent of the choice of local trivialization.

Again :

We know that the covariant derivative should be independent of coordinates.
It should transform like $D_{\mu'} s^{i'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \Lambda^{i'}_i D_\mu s^i$ if we choose a different local trivialization ($v^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} v^\mu$, $s^{i'} = \Lambda^{i'}_i s^i$).

Thus : $e_i = \Lambda^{i'}_i e_{i'}$

$$\cancel{\frac{\partial x^\mu}{\partial x^{\mu'}} \Lambda^{i'}_i \partial_\mu s^i} + \frac{\partial x^\mu}{\partial x^{\mu'}} \Lambda^{i'}_i A^i_{\mu j} s^j = \partial_{\mu'} s^{i'} + A^{i'}_{\mu' j'} s^{j'}$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu\left(\Lambda^{i'}_i s^i\right) + A^{i'}_{\mu' j'} \Lambda^{j'}_j s^j$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \left(\partial_\mu \Lambda^{i'}_j\right) s^j + \cancel{\frac{\partial x^\mu}{\partial x^{\mu'}} \Lambda^{i'}_i \partial_\mu s^i} + A^{i'}_{\mu' j'} \Lambda^{j'}_j s^j$$

$$\Rightarrow \quad A^{i'}_{\mu' k'} = A^i_{\mu j} \frac{\partial x^\mu}{\partial x^{\mu'}} (\Lambda^{-1})^j_{k'} \Lambda^{i'}_i - \frac{\partial x^\mu}{\partial x^{\mu'}} (\Lambda^{-1})^j_{k'} (\partial_\mu \Lambda^{i'}_j) .$$

So

$$\frac{d}{dt} v^{i'}(t) + A^{i'}_{\mu' k'} \gamma^{\mu'} v^{k'} = \frac{d}{dt}\left(\Lambda^{i'}_i v^i(t)\right) + A^i_{\mu j} \cancel{\frac{\partial x^\mu}{\partial x^{\mu'}}} \cancel{(\Lambda^{-1})^j_{k'}} \Lambda^{i'}_i \cancel{\frac{\partial x^{\mu'}}{\partial x^\mu}} \gamma^\mu \cancel{\Lambda^{k'}_k} v^k - \frac{\partial x^\mu}{\partial x^{\mu'}} (\Lambda^{-1})^j_{k'} (\partial_\mu \Lambda^{i'}_j) \frac{\partial x^{\mu'}}{\partial x^\mu} \gamma^\mu \Lambda^{k'}_k v^k$$

$$= \cancel{\left(\frac{d}{dt} \Lambda^{i'}_i\right) v^i(t)} + \Lambda^{i'}_i \frac{d}{dt} v^i(t) + A^i_{\mu k} \Lambda^{i'}_i \gamma^\mu v^k - \cancel{(\partial_\mu \Lambda^{i'}_k) \gamma^\mu v^k}$$

$$= \Lambda^{i'}_i \left[\frac{d}{dt} v^i(t) + A^i_{\mu k} \gamma^\mu v^k\right] e_i .$$

So the vector potential A transforms in exactly the right way to ensure independance of LT.

Excersice 89 :

$$u(t) = \sum_{n=0}^{\infty} \left((-1)^n \int_{t \geq t_1 \geq \dots \geq t_n \geq 0} A(\gamma'(t_1)) \cdots A(\gamma'(t_n))\, dt_n \cdots dt_1 \right) u$$

Define $\|T\| = \sup_{\|u\|=1} \|Tu\|$ on $\mathrm{End}(V)$ and let $K = \sup_{t \in [0,t]} \|A(\gamma'(t))\|$.

The nth term in the sum above has the norm :

$$\left\| (-1)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} A(\gamma'(t_1)) \cdots A(\gamma'(t_n))\, u \, dt_n \cdots dt_1 \right\|$$

$$\leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \| A(\gamma'(t_1)) \cdots A(\gamma'(t_n))\, u \| \, dt_n \cdots dt_1$$

$$\leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} K^n \|u\| \, dt_n \cdots dt_1$$

$$= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} K^n \|u\| \, t_{n-1} \, dt_{n-1} \cdots dt_1$$

$$= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-3}} K^n \|u\| \, \tfrac{1}{2} t_{n-2}^2 \, dt_{n-2} \cdots dt_1$$

$$= \dots$$

$$= K^n \|u\| t^n / n! \, .$$

So it has a converging majorante and thus converges !

We have shown that $\|u(t)\| \leq \sum_{n=0}^{\infty} K^n t^n \|u\| / n!$ and since the right side is differentiable, so is $u(t)$. Moreover $\|u(0)\| \leq \|u\| \Rightarrow u(0) = u$ because it can't be smaller.

$$\frac{d}{dt} u(t) = -A(\gamma'(t))u + A(\gamma'(t)) \int_0^t A(\gamma'(t_1)) u \, dt_1 - A(\gamma'(t)) \int_0^t \int_0^{t_1} A(\gamma'(t_1)) A(\gamma'(t_2)) u \, dt_2 \, dt_1 + \dots$$

$$= -A(\gamma'(t)) \left[u - \int_0^t A(\gamma'(t_1)) u \, dt_1 + \int_0^t \int_0^{t_1} A(\gamma'(t_1)) A(\gamma'(t_2)) u \, dt_2 \, dt_1 - \dots \right]$$

$$= -A(\gamma'(t))\, u(t) \, .$$

Excersice 90 :

We need to show that the holonomy depends only on the path and not the parametrization. This will hold if the covariant derivative is zero independent of the param.

Let's call the vector $u_\alpha(t)$ the vector in the fiber above $\alpha(t)$ and $u_\beta(t)$ the vector in the fiber above $\beta(t) = \alpha(f(t))$.

$$\Rightarrow D_{\beta'(t)} u_\beta(t) = D_{\beta'(t)} u_\alpha(f(t)) = \frac{d}{dt} u_\alpha(f(t)) + A(\beta'(t))\, u_\alpha(f(t))$$

$$= \frac{d}{ds} u_\alpha(s) \Big|_{s=f(t)} f'(t) + A\big(f'(t)\, \alpha'(f(t))\big)\, u_\alpha(f(t))$$

$$= f'(t) \left[\frac{d}{ds} u_\alpha(s) + A(\alpha'(s))\, u_\alpha(s) \right]_{s=f(t)}$$

$$= f'(t) \underbrace{D_{\alpha'(s)} u_\alpha(s)}_{=0} \, .$$

So we have shown that if u_α is parallel-transported along $\alpha(t)$, it is also parallel-transported along $\beta(t$

There is an even easier way to see this using the path-ordered exponential :

$$H(\gamma(t), D) = u(t) = P\, e^{-\int_0^t dt\, A(\gamma'(t))}$$

Choose $\gamma'(t) = \beta'(t) = f'(t)\,\alpha'(f(t)) \Rightarrow dt\, A(\beta'(t)) = dt\, f'(t)\, A(\alpha'(f(t))) \underset{\substack{\uparrow \\ s = f(t)}}{=} ds\, A(\alpha'(s))$

$$\Rightarrow H(\beta(t), D) = P\, e^{-\int_0^T dt\, A(\beta'(t))} = P\, e^{-\int_0^S ds\, A(\alpha'(s))} = H(\alpha(s), D) .$$

Excersice 91 :

$$H(\beta\alpha, D) = P\, e^{-\int_0^{T+S} dt\, A((\beta\alpha)'(t))} = P\, e^{-\int_0^T dt\, A(\beta'(t)) - \int_0^S ds\, A(\alpha'(s))}$$

$$= P\, e^{-\int_0^T dt\, A(\beta'(t))}\; P\, e^{-\int_0^S ds\, A(\alpha'(s))}$$

$$= H(\beta, D)\, H(\alpha, D) .$$

And $H(1_p, D) = P\, e^{-\int_0^1 dt\, A(1_p'(t))} = P\, e^0 = 1 .$

So it is clear that $H(1_q\alpha, D) = H(1_q, D)\, H(\alpha, D) = H(\alpha, D)$

$H(\alpha 1_p, D) = H(\alpha, D)\, H(1_p, D) = H(\alpha, D) .$

Excersice 92 :

The holonomy doesn't depend on the local trivialization, since the covariant derivative doesn't, so we just break up the path in the different local trivializations, and the result will be independent of the choices of local triv. :

$$H(\gamma, D') = H(\gamma_2, D')\, H(\gamma_1, D')$$

$$= g(\gamma_2(T))\, H(\gamma_2, D)\, g(\gamma_2(0))^{-1}\, g(\gamma_1(T))\, H(\gamma_1, D)\, g(\gamma_1(0))^{-1}$$

$$= g(\gamma_2(T))\, H(\gamma_2, D)\, H(\gamma_1, D)\, g(\gamma_1(0))^{-1}$$

$$= g(\gamma(T))\, H(\gamma, D)\, g(\gamma(0))^{-1} .$$

Excersice 93 :

D is a $\mathcal{G}$-connection when the components A_μ of the vector potential live in $\mathfrak{g}$, the Lie algebra of $\mathcal{G}$. Then, according to the path integral formula, the holonomy is an operator generated by elements of $\mathfrak{g}$, so it must live in the Lie group $\mathcal{G}$.

Excersice 94 :

$$[v, fw] = v(fw) - fw(v) = v(f)w + f vw - f wv = v(f)w + f[v,w]$$

Excersice 95 :

We use $H(\gamma, D) = P\, e^{-\oint dt\, A(\gamma'(t))}$. Now the path γ consists of four parts

$\gamma_1 : t \mapsto (x^\mu = t, x^\nu = \text{const})$

$\gamma_2 : t \mapsto (x^\mu = \text{const}, x^\nu = t)$

$\vdots$

$$\Rightarrow H(\gamma, D) = P\, e^{-\int_0^\varepsilon dt\, A_\mu - \int_0^\varepsilon dt\, A'_\nu + \int_0^\varepsilon dt\, A''_\mu + \int_0^\varepsilon dt\, A'''_\nu}$$

$$\approx 1 - \oint dt\, A(\gamma'(t)) + \frac{1}{2} P\left(\oint dt\, A(\gamma'(t))\right)^2 \longrightarrow = \int_0^1\int_0^{t_1} dt_2\, dt_1\, A(t_1)\, A(t_2)$$

$$= \int_0^\varepsilon dt\, A_\mu(\varepsilon, 0) + \int_0^\varepsilon dt\, A_\nu(\varepsilon, t) - \int_0^\varepsilon dt\, A_\mu(\varepsilon - t, \varepsilon) - \int_0^\varepsilon dt\, A_\nu(0, \varepsilon - t)$$

$$\approx \int_0^\varepsilon dt\, t A_\mu A_\mu + A_\nu(A_\mu + (t-\varepsilon)A_\nu) - (A_\mu + A_\nu - (t-\varepsilon)A_\mu)A_\mu - A_\nu(A_\nu - (t-\varepsilon)A_\nu)$$

$$\approx \int_0^\varepsilon dt\, [A_\mu + t\,\partial_\mu A_\mu + A_\nu + \varepsilon\,\partial_\mu A_\nu + t\,\partial_\nu A_\nu - A_\mu + t\,\partial_\mu A_\mu - \varepsilon\,\partial_\nu A_\mu - A_\nu + \partial_\nu A_\nu t] = \varepsilon^2(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= \cancel{\tfrac{1}{2}\varepsilon^2 A_\mu^2} + \varepsilon A_\nu A_\mu + \tfrac{1}{2}\varepsilon^2 \cancel{A_\nu^2} - \cancel{\varepsilon A_\mu^2} - A_\nu A_\mu \varepsilon + \cancel{\tfrac{1}{2}\varepsilon^2 A_\mu^2} - \cancel{\varepsilon A_\nu^2} + \tfrac{1}{2}\varepsilon^2 A_\nu^2$$

Exercise 96 :

We can choose the homotopy $\gamma_s(t) = (1-s)\gamma_0(t) + s\,\gamma_1(t)$.

Then we get $H(\gamma_s, D) = P\, e^{-\int_0^t dt\, A(\gamma_s'(t))} = P\, e^{-\int_0^t dt\, A(\gamma_0'(t)) + s\int_0^t dt\, [A(\gamma_0'(t)) - A(\gamma_1'(t))]}$

So that $\frac{d}{ds} H(\gamma_s, D) = \int_0^t dt\, [A(\gamma_0'(t)) - A(\gamma_1'(t))]\, H(\gamma_s, D)$.

From here it should follow that $\frac{d}{ds} H(\gamma_s, D) = 0$ and that $H(\gamma, D)$ doesn't depend on the path γ.

Moreover, we have shown that $H(\gamma, D) = 1 - \varepsilon^2 F_{\mu\nu}$ around a closed path.

In the case of a flat connection, we have $F_{\mu\nu} = 0$. So we can devide the loop into to arbitrary parts γ_0 and γ_1, and get $H(\gamma, D) = H(\gamma_0, D) H(\gamma_1^{-1}, D) = H(\gamma_0, D) H(\gamma_1, D)^{-1} = 1$.

It follows $H(\gamma_0, D) = H(\gamma_1, D)$.

Excersice 97 :

If M is 1-dimensional, we have only one coordinate, say x, and one associated coordinate vector field ∂_x, which is a basis for vector fields in an open set U.

Thus $u = f(x)\partial_x$ and $v = g(x)\partial_x$, so that $F(v, u) = f(x) g(x) F(\partial_x, \partial_x) = 0$, since F is antisymmetric.

Excersice 98 :

In exercise 72, we have shown that any section of $E \otimes E'$ can be written, not necessarily unique, as a local finite sum of sections of the form $s \otimes s'$, where $s \in \Gamma(E)$ and $s' \in \Gamma(E')$.

Apply this for $E \otimes \Lambda^p T^*M$ and get that any E-valued differential form can be written as $s \otimes \omega$, where s is a section of E and ω a differential form on M.

Exercise 99 :

Define the wedge product by the given formula, and then show that the resulting expression is basis independent and $C^\infty(M)$-linear in each factor.

Start from an E-valued form $\alpha = \sum_{j,k} b_{jk}\, s_j \otimes \omega_k = \sum_{j,k} c_{jk}\, s_j' \otimes \omega_k'$ for two bases $\{s_j\}, \{\omega_k\}$ and $\{s_j'\}, \{\omega_k'\}$. The bases are related as $s_j = T_j^k s_k'$, $\omega_j = R_j^k \omega_k'$.

Define $\alpha \wedge \mu = \sum_{j,k} b_{jk}\, s_j \otimes (\omega_k \wedge \mu)$ for μ an ordinary form.

Now change the basis

$$\alpha \wedge \mu = \sum_{j,k} b_{jk}\, T_j^l s_l' \otimes (R_k^m \omega_m' \wedge \mu) = \sum_{j,k} b_{jk}\, T_j^l R_k^m\, s_l' \otimes (\omega_m' \wedge \mu) = \sum_{l,m} c_{lm}\, s_l' \otimes (\omega_m' \wedge \mu)$$

which is what we would have obtained by defining the product in the other basis.

Thus the product is base-independent and $C^\infty(M)$-linear by construction.

Exercise 100 :

On the one hand we have defined $d_D s(v) = D_v s$. On the other hand, we have $d_D s = D_\mu s \otimes dx^\mu$ in local coordinates. To see that they are equivalent, evaluate it and get

$d_D s(v) = D_\mu s \otimes dx^\mu(v) = v(x^\mu) D_\mu s = v^\sigma \partial_\sigma(x^\mu) D_\mu s = v^\mu D_\mu s = D_v s$.

So they are equivalent.

Exercise 101 :

See 98 and 99. Everything is linear, so it is basis-independent.

Exersice 102 :

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]$$

$$= X[Y,Z] - [Y,Z]X + Y[Z,X] - [Z,X]Y + Z[X,Y] - [X,Y]Z$$

$$= \cancel{XYZ} - \cancel{XZY} - \cancel{YZX} + \cancel{ZYX} + \cancel{YZX} - \cancel{YXZ} - \cancel{ZXY} + \cancel{XZY} + \cancel{ZXY} - \cancel{ZYX} - \cancel{XYZ} + \cancel{YXZ}$$

$$= 0$$

Exersice 103 :

$$[D_v^*(\alpha\lambda)](s) = v(\alpha\lambda(s)) - \alpha\lambda D_v(s) = \alpha v(\lambda(s)) - \alpha\lambda D_v(s) = \alpha [D_v^*\lambda](s)$$

$$[D_v^*(\lambda+\rho)](s) = v((\lambda+\rho)(s)) - (\lambda+\rho)D_v(s) = [D_v^*\lambda](s) + [D_v^*\rho](s)$$

$$[D_v^*(f\lambda)](s) = v(f\lambda(s)) - f\lambda D_v(s) - v(f)\lambda(s) = f[D_v^*(\lambda)](s)$$

$$[D_{v+w}^*\lambda](s) = (v+w)(\lambda(s)) - \lambda D_{v+w}(s) = [D_v^*\lambda](s) + [D_w^*\lambda](s)$$

$$[D_{fv}^*\lambda](s) = (fv)(\lambda(s)) - \lambda D_{fv}(s) = f[D_v^*\lambda](s)$$

Exersice 104 :

Each entry of $(D\oplus D')_v(s,s') = (D_v s, D'_v s')$ is a connection, so the whole thing is obviously a connection.

Exersice 105 :

$$(D\otimes D')_v(\alpha s\otimes\beta s') = (D_v(\alpha s))\otimes\beta s' + \alpha s\otimes(D'_v(\beta s')) = \alpha\beta\,(D\otimes D')_v(s\otimes s')$$

$$(D\otimes D')_v((s+t)\otimes(s'+t')) - (D_v(s+t))\otimes(s'+t') + (s+t)\otimes(D'_v(s'+t')) = \ldots$$

$$(D\otimes D')_v(fs\otimes gs') = (D_v(fs))\otimes gs' + fs\otimes(D'_v(gs')) = fg\,(D\otimes D')_v(s\otimes s')$$

$$(D\otimes D')_{v+w}(s\otimes s') = (D_{v+w}s)\otimes s' + s\otimes(D'_{v+w}s') = (D\otimes D')_v(s\otimes s') + (D\otimes D')_w(s\otimes s')$$

$$(D\otimes D')_{fv}(s\otimes s') = (D_{fv}s)\otimes s' + s\otimes(D'_{fv}s') = f\,(D\otimes D')_v(s\otimes s')$$

Exersice 106 :

Since $\mathrm{End}(E) = E\otimes E^*$, define the connection on $\mathrm{End}(E)$ like in Ex. 105 and 103 :

$$\begin{aligned}(D\otimes D^*)_v\underbrace{(s\otimes\lambda)}_{=T}(t) &= [D_v s\otimes\lambda](t) + [s\otimes D_v^*\lambda](t)\\ &= D_v s\otimes\lambda(t) + s\otimes[v(\lambda(t)) - \lambda D_v t]\\ &= D_v s\otimes\lambda(t) + s\otimes v(\lambda(t)) - s\otimes\lambda D_v t\\ &= D_v(s\otimes\lambda(t)) - s\otimes\lambda D_v t\\ &=: D_v(Tt) - T(D_v t)\end{aligned}$$

Now define $(D\otimes D^*)_v(s\otimes\lambda)(t) =: (D_v T)(t)$ and get the wished result.

Exersice 107 :

In local coordinates we can write $\omega = T_I\otimes dx^I$ for the End(E)-valued p-form and $\mu = s_J\otimes dx^J$ for the E-valued form.

Use Ex. 106 and $d_D(s_I\otimes dx^I) = D_\mu s_I\otimes dx^\mu\wedge dx^I$ and $(T\otimes\omega)\wedge(s\otimes\mu) = T(s)\otimes(\omega\wedge\mu)$.

$$\begin{aligned}
d_D(\omega \wedge \mu) &= d_D\left([T_I \otimes dx^I] \wedge [s_J \otimes dx^J] \right) \\
&= d_D\left(T_I(s_J) \otimes (dx^I \wedge dx^J) \right) \\
&= d_D\left(T_I(s_J) \right) \wedge \left(dx^I \wedge dx^J \right) \\
&= D_\mu\left(T_I(s_J) \right) \otimes dx^\mu \wedge dx^I \wedge dx^J \\
&= \left[(D_\mu T_I) s_J + T_I (D_\mu s_J) \right] \otimes dx^\mu \wedge dx^I \wedge dx^J \\
&= (D_\mu T_I) s_J \otimes dx^\mu \wedge dx^I \wedge dx^J \; + \; T_I(D_\mu s_J) \otimes dx^\mu \wedge dx^I \wedge dx^J \\
&= \left[D_\mu T_I \otimes dx^\mu \wedge dx^I \right] \wedge \left[s_J \otimes dx^J \right] + \left[T_I \otimes dx^I \right] \wedge (-1)^p \left[D_\mu s_J \otimes dx^\mu \wedge dx^J \right] \\
&= d_D \omega \wedge \mu + (-1)^p \omega \wedge d_D \mu \, .
\end{aligned}$$

Exersice 108 :

$$\begin{aligned}
d_D F &= d_D\left(\tfrac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu \right) = \tfrac{1}{2} D_\lambda F_{\mu\nu} \otimes dx^\lambda \wedge dx^\mu \wedge dx^\nu \\
&= \frac{1}{3} \cdot \frac{1}{2} \left[D_\lambda F_{\mu\nu} + D_\lambda F_{\mu\nu} + D_\lambda F_{\mu\nu} \right] \otimes dx^\lambda \wedge dx^\mu \wedge dx^\nu \\
&= \frac{1}{3!} \left[D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} \right] \otimes dx^\lambda \wedge dx^\mu \wedge dx^\nu \, .
\end{aligned}$$

Exersice 109 :

I will do it examplarily for one path. It is completly anolog to Ex. 95.
Consider the path

$$\gamma_2^{-1}\gamma_1(t) = \begin{cases} (6t\varepsilon, 0, 0) \, , & 0 \le t \le 1/6 \\ (\varepsilon, \varepsilon(6t-1), 0) \, , & 1/6 \le t \le 2/6 \\ (\varepsilon, \varepsilon, \varepsilon(6t-2)) \, , & 2/6 \le t \le 3/6 \\ (\varepsilon(4-6t), \varepsilon, \varepsilon) \, , & 3/6 \le t \le 4/6 \\ (0, \varepsilon, \varepsilon(5-6t)) \, , & 4/6 \le t \le 5/6 \\ (0, \varepsilon(6-6t), 0) \, , & 5/6 \le t \le 1 \end{cases} \quad \Rightarrow \quad (\gamma_2^{-1}\gamma_1)'(t) = \begin{cases} 6\varepsilon \partial_\mu \, , & 0 \le t \le 1/6 \\ 6\varepsilon \partial_\nu \, , & 1/6 \le t \le 2/6 \\ 6\varepsilon \partial_\lambda \, , & 2/6 \le t \le 3/6 \\ -6\varepsilon \partial_\mu \, , & 3/6 \le t \le 4/6 \\ -6\varepsilon \partial_\lambda \, , & 4/6 \le t \le 5/6 \\ -6\varepsilon \partial_\nu \, , & 5/6 \le t \le 1 \end{cases}$$

$$\begin{aligned}
H(\gamma_2^{-1}\gamma_1, D) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} P\left(\int_0^1 ds \; A(\gamma'(s); \gamma(s)) \right)^n \\
&\approx 1 - \int_0^1 ds \; A(\gamma'(s); \gamma(s)) + \frac{1}{2} P\left(\int_0^1 ds \; A(\gamma'(s); \gamma(s)) \right)^2 - \frac{1}{3!} P\left(\int_0^1 ds \; A(\gamma'(s); \gamma(s)) \right)^3 + \dots
\end{aligned}$$

Let's concentrate on the first integral :

$$\begin{aligned}
\int_0^1 ds \; A(\gamma'(s); \gamma(s)) = &\int_0^{1/6} ds \; 6\varepsilon A_\mu(6s\varepsilon, 0, 0) + \int_{1/6}^{2/6} ds \; 6\varepsilon A_\nu(\varepsilon, \varepsilon(6s-1), 0) + \int_{2/6}^{3/6} ds \; 6\varepsilon A_\lambda(\varepsilon, \varepsilon, \varepsilon(6s-2)) \\
&- \int_{3/6}^{4/6} ds \; 6\varepsilon A_\mu(\varepsilon(4-6s), \varepsilon, \varepsilon) - \int_{4/6}^{5/6} ds \; 6\varepsilon A_\lambda(0, \varepsilon, \varepsilon(5-6s)) - \int_{5/6}^{1} ds \; 6\varepsilon A_\nu(0, \varepsilon(6-6s), 0)
\end{aligned}$$

Taylor expand the integrands :

$$A_\mu(6s\varepsilon, 0, 0) = A_\mu + 6s\varepsilon \partial_\mu A_\mu + \tfrac{1}{2} 6^2 s^2 \varepsilon^2 \partial_\mu \partial_\mu A_\mu + o(\varepsilon^3)$$

$$A_\nu(\varepsilon, \varepsilon(6s-1), 0) = A_\nu + \varepsilon \partial_\mu A_\nu + \varepsilon(6s-1)\partial_\nu A_\nu + \tfrac{1}{2}\varepsilon^2 \partial_\mu \partial_\mu A_\nu + \tfrac{1}{2}\varepsilon^2 (6s-1)^2 \partial_\nu \partial_\nu A_\nu + \varepsilon^2 (6s-1) \partial_\mu \partial_\nu A_\nu + o(\varepsilon^3)$$

$$A_\lambda(\varepsilon, \varepsilon, \varepsilon(6s-2)) = A_\lambda + \varepsilon \partial_\mu A_\lambda + \varepsilon \partial_\nu A_\lambda + \varepsilon(6s-2)\partial_\lambda A_\lambda + \tfrac{1}{2}\varepsilon^2 \partial_\mu \partial_\mu A_\lambda + \tfrac{1}{2}\varepsilon^2 \partial_\nu \partial_\nu A_\lambda + \tfrac{1}{2}\varepsilon^2 (6s-2)^2 \partial_\lambda \partial_\lambda A_\lambda + \varepsilon^2 \partial_\mu \partial_\nu A_\lambda + \varepsilon^2 (6s-2) [\partial_\mu \partial_\lambda A_\lambda + \partial_\nu \partial_\lambda A_\lambda]$$

$$A_\mu(\varepsilon(4-6s), \varepsilon, \varepsilon) = A_\mu + \varepsilon(4-6s)\partial_\mu A_\mu + \varepsilon \partial_\nu A_\mu + \varepsilon \partial_\lambda A_\mu + \tfrac{1}{2}\varepsilon^2 (4-6s)^2 \partial_\mu \partial_\mu A_\mu + \tfrac{1}{2}\varepsilon^2 \partial_\nu \partial_\nu A_\mu + \tfrac{1}{2}\varepsilon^2 \partial_\lambda \partial_\lambda A_\mu + \varepsilon^2 \partial_\nu \partial_\lambda A_\mu + \varepsilon^2 (4-6s) [\partial_\mu \partial_\nu A_\mu + \partial_\mu \partial_\lambda A_\mu]$$

$$A_\lambda(0, \varepsilon, \varepsilon(5-6s)) = A_\lambda + \varepsilon \partial_\nu A_\lambda + \varepsilon(5-6s)\partial_\lambda A_\lambda + \tfrac{1}{2}\varepsilon^2 \partial_\nu \partial_\nu A_\lambda + \tfrac{1}{2}\varepsilon^2 (5-6s)^2 \partial_\lambda \partial_\lambda A_\lambda + \varepsilon^2 (5-6s) \partial_\nu \partial_\lambda A_\lambda + o(\varepsilon^3)$$

$$A_\nu(0, \varepsilon(6-6s), 0) = A_\nu + \varepsilon(6-6s)\partial_\nu A_\nu + \tfrac{1}{2}\varepsilon^2 (6-6s)^2 \partial_\nu \partial_\nu A_\nu + o(\varepsilon^3)$$

$$\Rightarrow \int_0^1 ds \; A(\gamma'(s); \gamma(s)) = \varepsilon^2 \left(\partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu A_\lambda - \partial_\lambda A_\mu \right) + \varepsilon^3 \partial_\nu \left(\partial_\mu A_\lambda - \partial_\lambda A_\mu \right) + \dots ?$$

For the second integral :

$$\frac{1}{2} P\Big(\int_0^1 ds\ A(\gamma'(s);\gamma(s))\Big)^2 = \int_0^1 \int_0^{s_1} ds_2\, ds_1\ A(\gamma'(s_1);\gamma(s_1))\, A(\gamma'(s_2);\gamma(s_2))$$

Taylor expand the integrands again and calculate the inside integral :

$$\int_0^{s_1} ds_2\ A(\gamma'(s_2);\gamma(s_2)) = 6\varepsilon \begin{cases} s_1 A_\mu + 6\varepsilon \frac{1}{2} s_1^2 \partial_\mu A_\mu & 0 \le s_1 \le 1/6 \\ \frac{1}{6} A_\mu + \frac{1}{2}\cdot\frac{1}{6}\varepsilon\partial_\mu A_\mu + (s_1 - 1/6)[A_\nu + \varepsilon\partial_\mu A_\nu] + \frac{1}{2}\cdot\frac{1}{6}(6s_1-1)^2\varepsilon\partial_\nu A_\nu & 1/6 \le s_1 \le 2/6 \\ \frac{1}{6}(A_\mu + \frac{1}{2}\varepsilon\partial_\mu A_\mu + A_\nu + \varepsilon\partial_\mu A_\nu + \frac{1}{2}\varepsilon\partial_\nu A_\nu) + (s_1 - \frac{2}{6})[A_\lambda + \varepsilon\partial_\mu A_\lambda + \varepsilon\partial_\nu A_\lambda] + \frac{1}{2}\cdot\frac{1}{6}(6s_1-2)^2\varepsilon\partial_\lambda A_\lambda & 2/6 \le s_1 \le 3/6 \\ \frac{1}{6}(A_\mu + \frac{1}{2}\varepsilon\partial_\mu A_\mu + A_\nu + \varepsilon\partial_\mu A_\nu + \frac{1}{2}\varepsilon\partial_\nu A_\nu + A_\lambda + \varepsilon\partial_\mu A_\lambda + \varepsilon\partial_\nu A_\lambda + \frac{1}{2}\varepsilon\partial_\lambda A_\lambda) - (s_1 - \frac{1}{2})[A_\mu + \varepsilon\partial_\nu A_\mu + \varepsilon\partial_\lambda A_\mu] + \frac{1}{2}\cdot\frac{1}{6}(4-6s_1)^2\partial_\mu A_\mu\varepsilon & \\ \frac{1}{6}(A_\nu + \frac{1}{2}\varepsilon\partial_\mu A_\mu + \varepsilon\partial_\mu A_\nu + \frac{1}{2}\varepsilon\partial_\nu A_\nu + A_\lambda + \varepsilon\partial_\mu A_\lambda + \varepsilon\partial_\nu A_\lambda + \frac{1}{2}\varepsilon\partial_\lambda A_\lambda - \varepsilon\partial_\nu A_\mu - \varepsilon\partial_\lambda A_\mu) - (s_1 - \frac{4}{6})[A_\lambda + \varepsilon\partial_\nu A_\lambda] + \frac{1}{2}\cdot\frac{1}{6}(5-6s_1)^2\partial_\lambda A_\lambda & \\ \frac{1}{6}(A_\nu + \frac{1}{2}\varepsilon\partial_\mu A_\mu + \varepsilon\partial_\mu A_\nu + \frac{1}{2}\varepsilon\partial_\nu A_\nu + \varepsilon\partial_\mu A_\lambda + \frac{1}{2}\varepsilon\partial_\lambda A_\lambda - \varepsilon\partial_\nu A_\mu - \varepsilon\partial_\lambda A_\mu) - (s_1 - \frac{5}{6})A_\nu + \frac{1}{2}\cdot\frac{1}{6}\varepsilon(6-6s_1)^2\partial_\nu A_\nu & \end{cases}$$

Inserting this in the s_1 integral gives

$$\begin{aligned} \int_0^1 ds_1 \int_0^{s_1} ds_2\ A(s_1)A(s_2) = 6^2\varepsilon^2\Big[& \int_0^{1/6} ds\ (A_\mu + 6s\varepsilon\partial_\mu A_\mu)(sA_\mu + 6\varepsilon\tfrac{1}{2}s^2\partial_\mu A_\mu) \\ & + \int_{1/6}^{2/6} ds\ (A_\nu + \varepsilon\partial_\mu A_\nu + \varepsilon(6s-1)\partial_\nu A_\nu)\big(\tfrac{1}{6}A_\mu + \tfrac{1}{2}\cdot\tfrac{1}{6}\varepsilon\partial_\mu A_\mu + (s-\tfrac{1}{6})[A_\nu + \varepsilon\partial_\mu A_\nu] + \tfrac{1}{2}\cdot\tfrac{1}{6}(6s-1)^2\varepsilon\partial_\nu A_\nu\big) \\ & + \int_{2/6}^{3/6} ds\ (A_\lambda + \varepsilon\partial_\mu A_\lambda + \varepsilon\partial_\nu A_\lambda + \varepsilon(6s-2)\partial_\lambda A_\lambda)\big(\tfrac{1}{6}[A_\mu + \tfrac{1}{2}\varepsilon\partial_\mu A_\mu + A_\nu + \varepsilon\partial_\mu A_\nu + \tfrac{1}{2}\varepsilon\partial_\nu A_\nu] + (s-\tfrac{2}{6})[A_\lambda + \varepsilon\partial_\mu A_\lambda + \varepsilon\partial_\nu A_\lambda] + \tfrac{1}{2}\cdot\tfrac{1}{6}(6s-2)^2\varepsilon\partial_\lambda A_\lambda\big) \\ & - \int_{3/6}^{4/6} ds\ (A_\mu + \varepsilon(4-6s)\partial_\mu A_\mu + \varepsilon\partial_\nu A_\mu + \varepsilon\partial_\lambda A_\mu)\big(\tfrac{1}{6}[A_\mu + \tfrac{1}{2}\varepsilon\partial_\mu A_\mu + A_\nu + \varepsilon\partial_\mu A_\nu + \tfrac{1}{2}\varepsilon\partial_\nu A_\nu + A_\lambda + \varepsilon\partial_\mu A_\lambda + \varepsilon\partial_\nu A_\lambda + \tfrac{1}{2}\varepsilon\partial_\lambda A_\lambda] - (s-\tfrac{1}{2})[A_\mu + \varepsilon\partial_\nu A_\mu + \varepsilon\partial_\lambda A_\mu] \\ & \qquad + \tfrac{1}{2}\cdot\tfrac{1}{6}(4-6s)^2\varepsilon\partial_\mu A_\mu\big) \\ & - \int_{4/6}^{5/6} ds\ (A_\lambda + \varepsilon(5-6s)\partial_\lambda A_\lambda + \varepsilon\partial_\nu A_\lambda)\big(\tfrac{1}{6}[A_\nu + \tfrac{1}{2}\varepsilon\partial_\mu A_\mu + \varepsilon\partial_\mu A_\nu + \tfrac{1}{2}\varepsilon\partial_\nu A_\nu + A_\lambda + \varepsilon\partial_\mu A_\lambda + \varepsilon\partial_\nu A_\lambda + \tfrac{1}{2}\varepsilon\partial_\lambda A_\lambda - \varepsilon\partial_\nu A_\mu - \varepsilon\partial_\lambda A_\mu] - (s-\tfrac{4}{6})[A_\lambda + \varepsilon\partial_\nu A_\lambda] \\ & \qquad + \tfrac{1}{2}\cdot\tfrac{1}{6}\varepsilon(5-6s)^2\partial_\lambda A_\lambda\big) \\ & - \int_{5/6}^{1} ds\ (A_\nu + \varepsilon(6-6s)\partial_\nu A_\nu)\big(\tfrac{1}{6}[A_\nu + \tfrac{1}{2}\varepsilon\partial_\mu A_\mu + \varepsilon\partial_\mu A_\nu + \tfrac{1}{2}\varepsilon\partial_\nu A_\nu + \varepsilon\partial_\mu A_\lambda + \tfrac{1}{2}\varepsilon\partial_\lambda A_\lambda - \varepsilon\partial_\nu A_\mu - \varepsilon\partial_\lambda A_\mu] - (s-\tfrac{5}{6})A_\nu + \tfrac{1}{2}\cdot\tfrac{1}{6}\varepsilon(6-6s)^2\partial_\nu A_\nu\big)\Big] \end{aligned}$$

$$\begin{aligned} = 6^2\varepsilon^2\Big[& \int_0^{1/6} ds\ (s\cancel{A_\mu A_\mu} + \cancel{6\varepsilon\tfrac{1}{2}s^2A_\mu\partial_\mu A_\mu} + \cancel{6s\varepsilon\partial_\mu A_\mu A_\mu}) \\ & + \int_{1/6}^{2/6} ds\ \big(\tfrac{1}{6}\underline{A_\nu A_\mu} + \cancel{\tfrac{1}{2}\cdot\tfrac{1}{6}\varepsilon A_\nu\partial_\mu A_\mu} + (s-\tfrac{1}{6})A_\nu[\cancel{A_\nu} + \varepsilon\partial_\mu A_\nu] + \tfrac{1}{2}\cdot\cancel{\tfrac{1}{6}(6s-1)^2\varepsilon A_\nu\partial_\nu A_\nu} + \tfrac{1}{6}\varepsilon\partial_\mu A_\nu A_\mu + \varepsilon(6s-1)\tfrac{1}{6}A_\mu\partial_\nu A_\nu\big) \\ & + \int_{2/6}^{3/6} ds\ \big(\tfrac{1}{6}[\underline{A_\lambda A_\mu} + \cancel{\tfrac{1}{2}\varepsilon A_\lambda\partial_\mu A_\mu} + \cancel{A_\lambda A_\nu} + \cancel{\varepsilon A_\lambda\partial_\mu A_\nu} + \cancel{\tfrac{1}{2}\varepsilon A_\lambda\partial_\nu A_\nu}] + (s-\tfrac{2}{6})[\cancel{A_\lambda A_\lambda} + \varepsilon A_\lambda\partial_\mu A_\lambda + \varepsilon A_\lambda\partial_\nu A_\lambda] + \tfrac{1}{2}\cdot\cancel{\tfrac{1}{6}(6s-2)^2\varepsilon A_\lambda\partial_\lambda A_\lambda} \\ & \qquad + \tfrac{1}{6}A_\mu\varepsilon\partial_\mu A_\lambda + \underline{\varepsilon A_\mu\tfrac{1}{6}\partial_\nu A_\lambda} + \tfrac{1}{6}\varepsilon(6s-2)A_\mu\partial_\lambda A_\lambda + \tfrac{1}{6}A_\nu\varepsilon\partial_\mu A_\lambda + \varepsilon A_\nu\tfrac{1}{6}\partial_\nu A_\lambda + \tfrac{1}{6}\varepsilon(6s-2)A_\nu\partial_\lambda A_\lambda + (s-2/6)A_\lambda\varepsilon\partial_\mu A_\lambda + \varepsilon(s-2/6)A_\lambda\partial_\nu A_\lambda \\ & \qquad + \varepsilon\cancel{(s-\tfrac{2}{6})(6s-2)A_\lambda\partial_\lambda A_\lambda}\big) \\ & - \int_{3/6}^{4/6} ds\ \big(\tfrac{1}{6}[\cancel{A_\mu A_\mu} + \cancel{\tfrac{1}{2}\varepsilon A_\mu\partial_\mu A_\mu} + \underline{A_\mu A_\nu} + \varepsilon A_\mu\partial_\mu A_\nu + \tfrac{1}{2}\varepsilon A_\mu\partial_\nu A_\nu + \underline{A_\mu A_\lambda} + \varepsilon\partial_\mu A_\lambda A_\mu + \underline{\varepsilon A_\mu\partial_\nu A_\lambda} + \tfrac{1}{2}\varepsilon A_\mu\partial_\lambda A_\lambda] - (s-\tfrac{1}{2})[\cancel{A_\mu A_\mu} + \varepsilon A_\mu\partial_\nu A_\mu \\ & \qquad + \varepsilon A_\mu\partial_\lambda A_\mu] + \cancel{\tfrac{1}{2}\cdot\tfrac{1}{6}(4-6s)^2\varepsilon A_\mu\partial_\mu A_\mu} + \cancel{\tfrac{1}{6}A_\mu\varepsilon(4-6s)\partial_\mu A_\mu} + \varepsilon\tfrac{1}{6}A_\mu\partial_\nu A_\mu + \tfrac{1}{6}\varepsilon A_\mu\partial_\lambda A_\mu + \tfrac{1}{6}\varepsilon(4-6s)A_\nu\partial_\mu A_\mu + \tfrac{1}{6}\varepsilon A_\nu\partial_\nu A_\mu + \tfrac{1}{6}\varepsilon A_\nu\partial_\lambda A_\mu \\ & \qquad + \tfrac{1}{6}\varepsilon(4-6s)A_\lambda\partial_\mu A_\mu + \underline{\tfrac{1}{6}\varepsilon A_\lambda\partial_\nu A_\mu} + \tfrac{1}{6}\varepsilon A_\lambda\partial_\lambda A_\mu - \cancel{(s-\tfrac{1}{2})A_\mu\varepsilon(4-6s)\partial_\mu A_\mu} - (s-\tfrac{1}{2})A_\mu\varepsilon\partial_\nu A_\mu - (s-\tfrac{1}{2})\varepsilon A_\mu\partial_\lambda A_\mu\big) \\ & - \int_{4/6}^{5/6} ds\ \big(\tfrac{1}{6}[\cancel{A_\lambda A_\nu} + \cancel{\tfrac{1}{2}\varepsilon A_\lambda\partial_\mu A_\mu} + \cancel{\varepsilon A_\lambda\partial_\mu A_\nu} + \cancel{\tfrac{1}{2}\varepsilon A_\lambda\partial_\nu A_\nu} + \cancel{A_\lambda A_\lambda} + \varepsilon A_\lambda\partial_\mu A_\lambda + \varepsilon A_\lambda\partial_\nu A_\lambda + \cancel{\tfrac{1}{2}\varepsilon A_\lambda\partial_\lambda A_\lambda} - \underline{\varepsilon A_\lambda\partial_\nu A_\mu} - \varepsilon A_\lambda\partial_\lambda A_\mu] - \cancel{(s-\tfrac{4}{6})A_\lambda A_\lambda} \\ & \qquad - (s-\tfrac{4}{6})\varepsilon A_\lambda\partial_\nu A_\lambda + \cancel{\tfrac{1}{2}\cdot\tfrac{1}{6}\varepsilon(5-6s)^2A_\lambda\partial_\lambda A_\lambda} + \tfrac{1}{6}A_\nu\varepsilon(5-6s)\partial_\lambda A_\lambda + \tfrac{1}{6}\varepsilon A_\nu\partial_\nu A_\lambda + \cancel{\tfrac{1}{6}\cdot\tfrac{1}{2}\varepsilon(5-6s)A_\lambda\partial_\lambda A_\lambda} + \tfrac{1}{6}\varepsilon A_\lambda\partial_\nu A_\lambda - \cancel{(s-\tfrac{4}{6})A_\lambda\varepsilon(5-6s)\partial_\lambda A_\lambda} \\ & \qquad - (s-\tfrac{4}{6})A_\lambda\varepsilon\partial_\nu A_\lambda\big) \\ & - \int_{5/6}^{1} ds\ \big(\tfrac{1}{6}[\cancel{A_\nu A_\nu} + \cancel{\tfrac{1}{2}\varepsilon A_\nu\partial_\mu A_\mu} + \varepsilon A_\nu\partial_\mu A_\nu + \cancel{\tfrac{1}{2}\varepsilon A_\nu\partial_\nu A_\nu} + \underline{\varepsilon A_\nu\partial_\mu A_\lambda} + \tfrac{1}{2}\varepsilon A_\nu\partial_\lambda A_\lambda - \varepsilon A_\nu\partial_\nu A_\mu - \underline{\varepsilon A_\nu\partial_\lambda A_\mu}] - \cancel{(s-\tfrac{5}{6})A_\nu A_\nu} + \cancel{\tfrac{1}{2}\tfrac{1}{6}(6-6s)^2A_\nu\partial_\nu A_\nu} \\ & \qquad + \cancel{\tfrac{1}{6}A_\nu\varepsilon(6-6s)\partial_\nu A_\nu} - \cancel{(s-\tfrac{5}{6})A_\nu\varepsilon(6-6s)\partial_\nu A_\nu}\big) \end{aligned}$$

$$= \varepsilon^2\big([A_\nu, A_\mu] + [A_\lambda, A_\mu]\big) + \varepsilon^3\big(\partial_\nu A_\lambda A_\mu + A_\lambda\partial_\nu A_\mu - A_\mu\partial_\nu A_\lambda - \partial_\nu A_\mu A_\lambda + A_\nu\partial_\lambda A_\mu - A_\nu\partial_\mu A_\lambda\big) + \dots ?$$

$$= \varepsilon^2\big([A_\nu, A_\mu] + [A_\lambda, A_\mu]\big) + \varepsilon^3\big(\partial_\nu[A_\lambda, A_\mu] + A_\nu(\partial_\lambda A_\mu - \partial_\mu A_\lambda)\big)$$ The third integral will give us the rest for the

Exercise 110 :

Analog to Ex. 101 .

Exercise 111 :

Analog to Ex. 107 , just do the calculation in local coordinates .

Exercise 112 :

- $[\omega,\mu] = \omega\wedge\mu - (-1)^{pq}\mu\wedge\omega = -(-1)^{pq}\mu\wedge\omega + (-1)^{pq}(-1)^{pq}\omega\wedge\mu = -(-1)^{pq}[\mu,\omega]$

- $[\omega,[\mu,\eta]] + (-1)^{p(q+r)}[\mu,[\eta,\omega]] + (-1)^{r(p+q)}[\eta,[\omega,\mu]]$

$$= \omega\wedge[\mu,\eta] - (-1)^{p(q+r)}[\mu,\eta]\wedge\omega + (-1)^{p(q+r)}\mu\wedge[\eta,\omega] - (-1)^{p(q+r)+q(p+r)}[\eta,\omega]\wedge\mu$$
$$+ (-1)^{r(p+q)}\eta\wedge[\omega,\mu] - (-1)^{r(p+q)+r(p+q)}[\omega,\mu]\wedge\eta$$

$$= \cancel{\omega\wedge\mu\wedge\eta} - (-1)^{qr}\cancel{\omega\wedge\eta\wedge\mu} - (-1)^{p(q+r)}\cancel{\mu\wedge\eta\wedge\omega} + (-1)^{p(q+r)}\cancel{\eta\wedge\mu\wedge\omega}(-1)^{qr} + (-1)^{p(q+r)}\cancel{\mu\wedge\eta\wedge\omega}$$
$$- (-1)^{p(q+r)}(-1)^{pr}\cancel{\mu\wedge\omega\wedge\eta} - (-1)^{r(p+q)}\cancel{\eta\wedge\omega\wedge\mu} + (-1)^{r(p+q)+rp}\cancel{\omega\wedge\eta\wedge\mu} + (-1)^{r(p+q)}\cancel{\eta\wedge\omega\wedge\mu}$$
$$- (-1)^{r(p+q)+pq}\cancel{\eta\wedge\mu\wedge\omega} - \cancel{\omega\wedge\mu\wedge\eta} + (-1)^{pq}\cancel{\mu\wedge\omega\wedge\eta}$$

$$= 0$$

- If A is an $End(E)$-valued form , we can write for example $A = A_\mu \otimes dx^\mu$

$$\Rightarrow A\wedge A = (A_\mu\otimes dx^\mu)\wedge(A_\nu\otimes dx^\nu) = A_\mu A_\nu \otimes (dx^\mu\wedge dx^\nu) = \tfrac{1}{2}[A_\mu,A_\nu]\otimes dx^\mu\wedge dx^\nu$$

And the commutator doesn't have to be zero !

- But $[A, A\wedge A] = [A_\lambda\otimes dx^\lambda, \tfrac{1}{2}[A_\mu,A_\nu]\otimes dx^\mu\wedge dx^\nu] = \tfrac{1}{2}[A_\lambda,[A_\mu,A_\nu]]\otimes dx^\lambda\wedge dx^\mu\wedge dx^\nu$

$$= \frac{1}{3!}\Big([A_\lambda,[A_\mu,A_\nu]] + [A_\mu,[A_\nu,A_\lambda]] + [A_\nu,[A_\lambda,A_\mu]]\Big)\otimes dx^\lambda\wedge dx^\mu\wedge dx^\nu$$
$$= 0$$

Because of the Jacobi identity !

Checking Yang-Mills-equation :

$F = B + E\wedge dt$

$B = \frac{1}{2}\varepsilon_{ijk} B^i dx^j\wedge dx^k$

$E = E_i dx^i$

$* F = *B + *(E\wedge dt) = *_s E - *_s B\wedge dt$

$$* d_D * F = * d_D(*_s E - *_s B\wedge dt) = *\big(dt\wedge D_t *_s E + d_s *_s E - d_s *_s B\wedge dt\big) = -*_s D_t *_s E - *_s d_s *_s E\wedge dt + *_s d_s *_s B$$
$$= -D_t E + *_s d_s *_s B - *_s d_s *_s E\wedge dt$$
$$= j - \rho\, dt$$

$F = \frac{1}{2}F_{\mu\nu} dx^\mu\wedge dx^\nu = \frac{1}{2}\varepsilon_{ijk}B^i dx^j\wedge dx^k + E_i dx^i\wedge dx^0$

$\Rightarrow E_i = F_{i0} = -\partial_t A_i$ (in temporal gauge $A_0 = 0$)

$d_D\eta = d\eta + [A,\eta]$

$B^i = \varepsilon^{ijk} F_{jk} = \varepsilon^{ijk}(\partial_j A_k - \partial_k A_j + [A_j, A_k])$

① $d_s B = 0$

$\partial^i B_i + [A^i, B_i] = 0$

② $D_t B + d_s E = 0$

$\partial_t B^i + \varepsilon^{ijk}(\partial_j E_k + [A_j, E_k]) = 0$

③ $*_s d_s *_s E = \rho$

$\partial^i E_i + [A^i, E_i] = \rho$

④ $-D_t E + *_s d_s *_s B = j$

$-\partial_t E^i + \varepsilon^{ijk}(\partial_j B_k + [A_j, B_k]) = j^i$

Exercise 113 :

$$d_{D'}\eta = D'_\mu \eta_I \otimes dx^\mu \wedge dx^I = g\, D_\mu (g^{-1}\eta_I) \otimes dx^\mu \wedge dx^I = g\, d_D (g^{-1}\eta) .$$

Exercise 114 :

$$\begin{aligned} D'_\nu T = [D'_\nu , T] = [g D_\nu g^{-1} , T] = g D_\nu g^{-1} T - T g D_\nu g^{-1} &= g D_\nu g^{-1} T g g^{-1} - g g^{-1} T g D_\nu g^{-1} \\ &= g [D_\nu , g^{-1} T g] g^{-1} , \\ &= g D_\nu (g^{-1} T g) g^{-1} \\ &= Ad(g)\, D_\nu (Ad(g^{-1}) T) . \end{aligned}$$

Exercise 115 :

$$\begin{aligned} d_{D'}\eta = [D'_\mu , \eta_I] \otimes dx^\mu \wedge dx^I &= g [D_\mu , g^{-1} \eta_I g] g^{-1} \otimes dx^\mu \wedge dx^I \\ &= g\, d_D (g^{-1} \eta g) g^{-1} \\ &= Ad(g)\, d_D (Ad(g^{-1}) \eta) . \end{aligned}$$

Exercise 116 :

Quick and dirty :

$$\begin{aligned} F = \tfrac{1}{2} F_{\mu\nu} \otimes dx^\mu \wedge dx^\nu &= \tfrac{1}{2} [D_\mu , D_\nu] \otimes dx^\mu \wedge dx^\nu \\ &= \tfrac{1}{2} [D^\circ_\mu + A_\mu , D^\circ_\nu + A_\nu] \otimes dx^\mu \wedge dx^\nu \\ &= \tfrac{1}{2} \Big([D^\circ_\mu , D^\circ_\nu] + [D^\circ_\mu , A_\nu] + [A_\mu , D^\circ_\nu] + [A_\mu , A_\nu] \Big) \otimes dx^\mu \wedge dx^\nu \\ &= \tfrac{1}{2} [D^\circ_\mu , D^\circ_\nu] \otimes dx^\mu \wedge dx^\nu + [D^\circ_\mu , A_\nu] \otimes dx^\mu \wedge dx^\nu + \tfrac{1}{2} [A_\mu , A_\nu] \otimes dx^\mu \wedge dx^\nu \\ &= F_\circ + dA + A \wedge A . \end{aligned}$$

Exercise 117 :

Write the End(E)-valued p-form $\omega = T \otimes \omega'$ and the q-form $\mu = S \otimes \mu'$.

Then $tr(\omega \wedge \mu) = tr(TS \otimes (\omega' \wedge \mu')) = tr(TS)\, \omega' \wedge \mu' = (-1)^{pq}\, tr(ST)\, \mu' \wedge \omega'$

Because $tr(TS) = tr(ST)$ and $\omega' \wedge \mu' = (-1)^{pq} \mu' \wedge \omega'$. $\qquad = (-1)^{pq}\, tr(\mu \wedge \omega)$.

Since we know that $[\omega , \mu] = \omega \wedge \mu - (-1)^{pq} \mu \wedge \omega$, it is

$$tr([\omega , \mu]) = tr(\omega \wedge \mu) - (-1)^{pq}\, tr(\mu \wedge \omega) = tr(\omega \wedge \mu) - (-1)^{2pq}\, tr(\omega \wedge \mu) = 0 .$$

Exercise 118 :

With the identity $d_D \omega = d\omega + [A , \omega]$, we get

$$tr(d_D \omega) = tr(d\omega) + tr([A , \omega]) = tr(d\omega) = d\, tr(\omega)$$

with the last exercise and the fact that d and tr are linear operators.

Exercise 119 :

$$\left.\begin{aligned} &\int_M tr(d_D(\omega \wedge \mu)) = \int_M tr(d_D \omega \wedge \mu) + (-1)^p \int_M tr(\omega \wedge d_D \mu) \\ &\text{But also } \int_M tr(d_D(\omega \wedge \mu)) = \int_M d\, tr(\omega \wedge \mu) = \int tr(\omega \wedge \mu) \overset{?}{=} 0 \end{aligned}\right\} \int_M tr(d_D \omega \wedge \mu) = (-1)^{p+1} \int_M tr(\omega \wedge d_D \mu)$$

Moreover $\int_M tr(\omega \wedge * \mu) = \int_M tr(\langle \omega, \mu \rangle) \, vol = \int_M tr(\langle \mu, \omega \rangle) \, vol = \int_M tr(\mu \wedge * \omega)$.

Exercise 120 :

The answer is already given in the exercise.
The integral in $S_{YM}(A)$ may not converge but $\delta S_{YM}(A) = \int_M tr(\delta A \wedge d_D * F)$ has a δA and if we restrict ourselves to variations that vanish outside some compact subset of M, this integral converges.

Exercise 121 :

Analise
$$\delta S = -\frac{1}{2} \delta \int_M F \wedge * F = -\frac{1}{2} \int_M (\delta F \wedge * F + F \wedge * \delta F) = -\int_M \delta F \wedge * F$$
$$= -\int_M \delta dA \wedge * F = -\int_M d \delta A \wedge * F = -\int_M \delta A \wedge d * F .$$

The integrand vanishes for an arbitrary variation δA if and only if $d * F = 0$.
When $M = \mathbb{R} \times S$ with the metric $dt^2 - {}^3g$, then we have $-F \wedge * F = -\langle F, F \rangle \, vol$.
And $\langle F, F \rangle = \langle B + E \wedge dt, B + E \wedge dt \rangle = \langle B, B \rangle + \langle E \wedge dt, E \wedge dt \rangle = \langle B, B \rangle - \langle E, E \rangle$.
So we get $-F \wedge * F = (\langle E, E \rangle - \langle B, B \rangle) \, vol$.
For the Yang-Mills Lagrangian this generalizes to $tr(F \wedge * F) = tr(\langle B, B \rangle) - tr(\langle E, E \rangle)$, where $\langle \cdot, \cdot \rangle$ denotes a matrix product now.

Exercise 122 :

If one has shown that $tr(F) = iB$, one can use the argument for charge quantization with $qm/\hbar = 2\pi N$ where $m = \int_{S^2} B$ and find that the first Chern class is integral :
Let $q/\hbar = 1$ $\Rightarrow$ $m = \int_{S^2} B = \frac{1}{i} \int_{S^2} tr(F) = 2\pi N$ $\Rightarrow$ $\frac{i}{2\pi} \int_{S^2} tr(F) = -N$.

Exercise 123 :

$$tr(F^k) = \int_0^1 \frac{d}{ds} tr(F_s^k) \, ds = k \int_0^1 tr\left(\frac{dF_s}{ds} \wedge F_s^{k-1} \right) ds = k \, d \int_0^1 tr(A \wedge F_s^{k-1}) \, ds$$
$$= k \, d \int_0^1 tr\left(A \wedge \sum_{i=0}^{k-1} \binom{k-1}{i} (s \, dA)^{k-1-i} \wedge (s^2 A \wedge A)^i \right) ds$$
$$= k \, d \int_0^1 tr\left(\sum_{i=0}^{k-1} s^{k-1+i} A \wedge \binom{k-1}{i} dA^{k-1-i} \wedge A^{2i} \right) ds$$
$$= d \, tr\left(\sum_{i=0}^{k-1} \frac{k}{k+i} \binom{k-1}{i} A \wedge dA^{k-1-i} \wedge A^{2i} \right).$$

Exercise 124 :

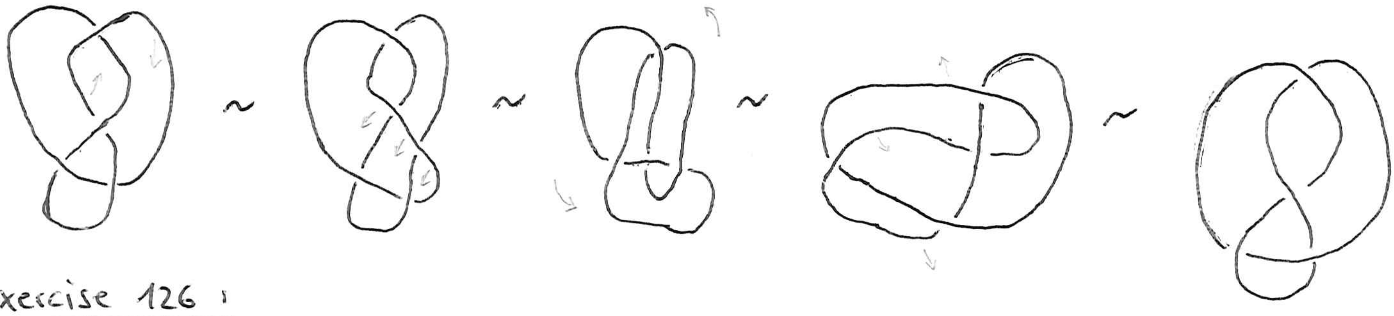
$$\frac{d}{ds} S_{CS}(A_s) \Big|_{s=0} = \int_S \frac{d}{ds} tr\left(A_s \wedge dA_s + \frac{2}{3} A_s \wedge A_s \wedge A_s \right) \Big|_{s=0}$$
$$= 2 \int_S tr\left(\frac{d}{ds} A_s \wedge dA_s + A_s \wedge A_s \wedge \frac{d}{ds} A_s \right) \Big|_{s=0}$$
$$= 2 \int_S tr\big(([T, A] - dT) \wedge dA + A \wedge A \wedge ([T, A] - dT) \big)$$
$$= 2 \int_S tr\big([T, A] \wedge dA + A \wedge A \wedge ([T, A] - dT) \big).$$

$dA_s |_{s=0} = dA$

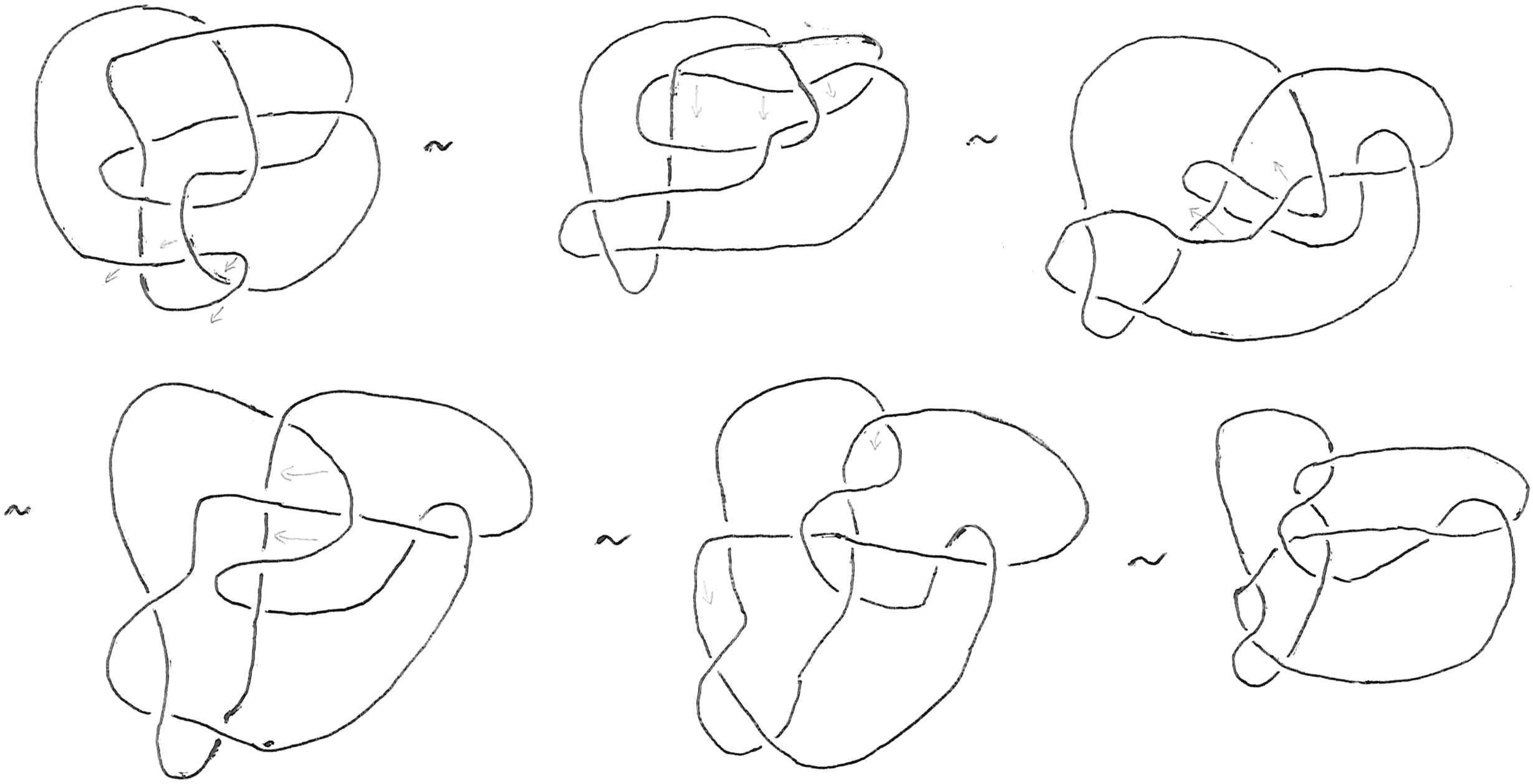
$A_s |_{s=0} = A$

$\int_S tr(dT \wedge dA) = 0$

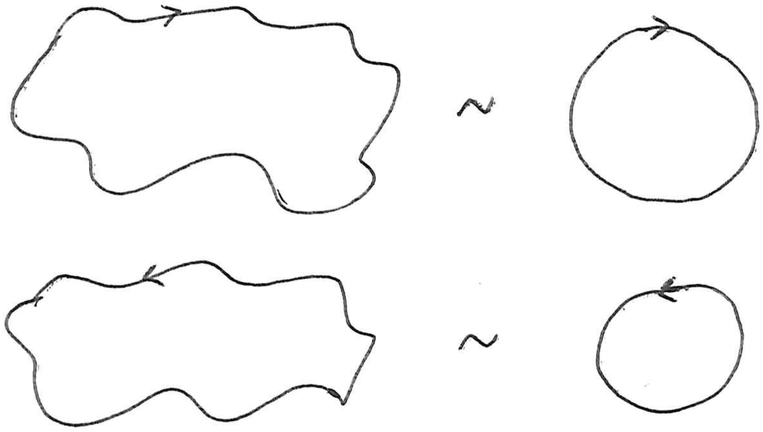
Exercise 125 :



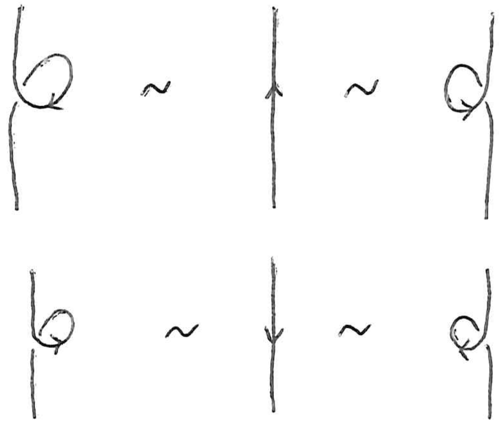
Exercise 126 :



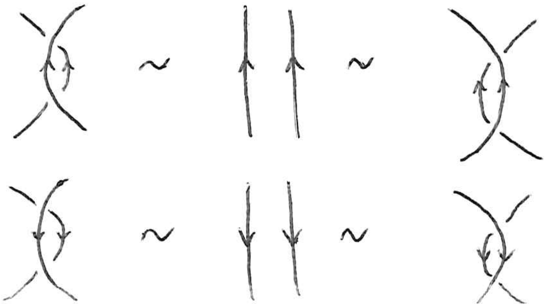
Exercise 127 :

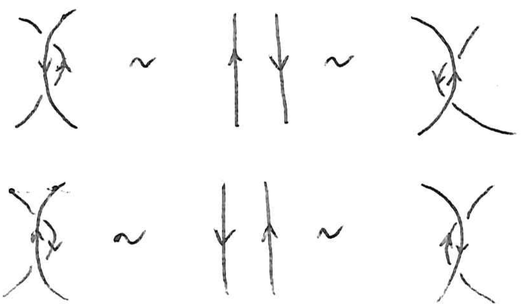


Zeroth Reidemeister move

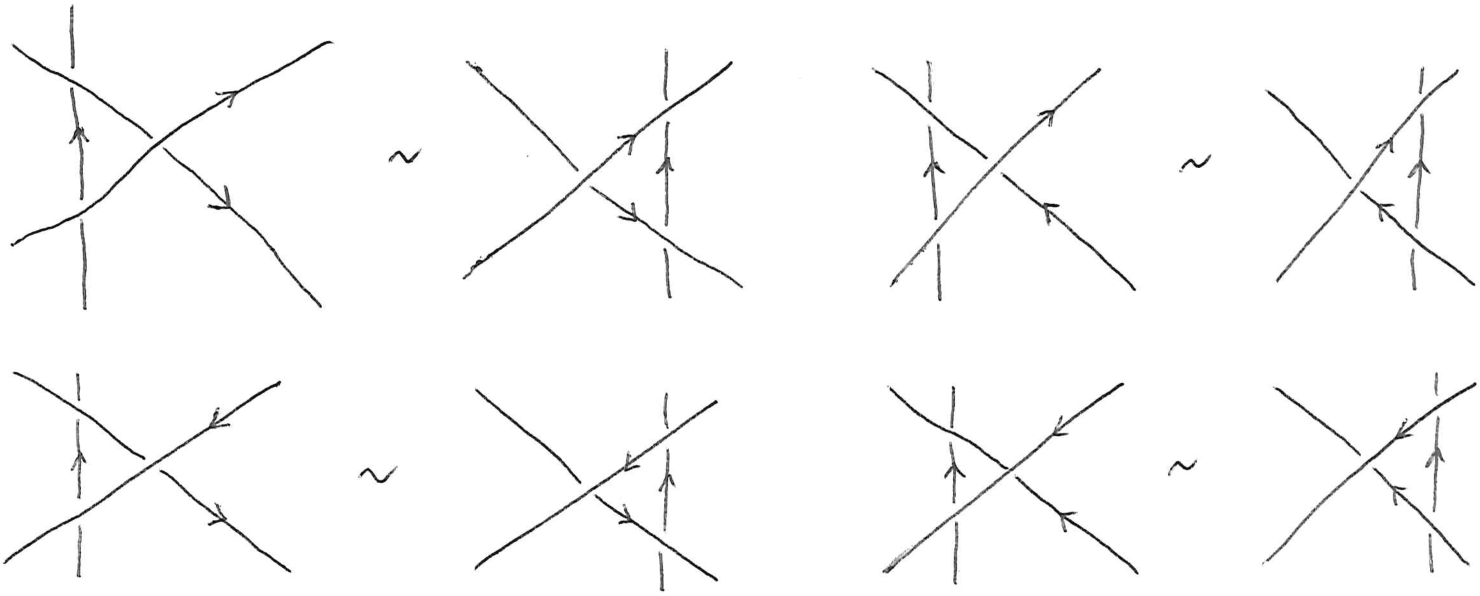


First Reidemeister move





Second Reidemeister move



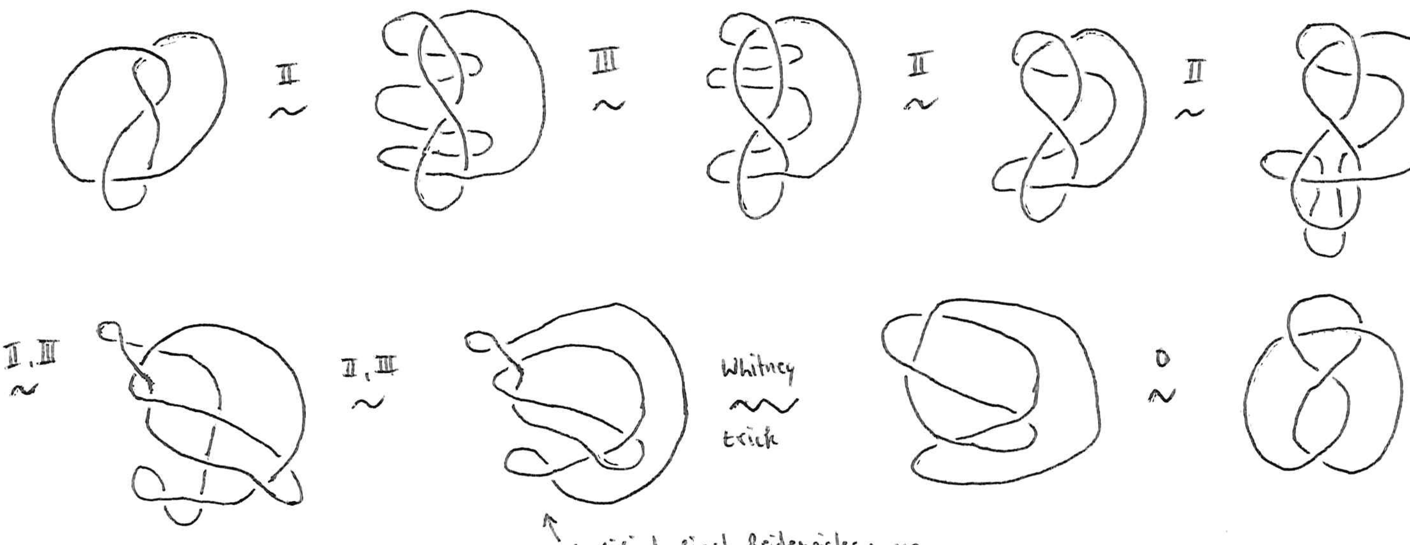
And the same with ↓, so we have 8 versions all together !

Third Reidemeister move

Exercise 128 :

Take a little piece of ribbon and check that it can be fold like this :

Exercise 129 :



Modified first Reidemeister move
with one twist to make them opposite

Exercise 130 :

The writhe is invariant under Reidemeister moves 0, II and III for the same reasons as for the linking number. Move 0 does not create any new crossing, so it preserves the writhe. Move II creates or destroys a pair of crossings, but with opposite handedness, so the writhe is unaffected. And also Move III does not change the number of crossings or their handedness. Finally, Move I` does not change the number of crossings and preserves the handedness !

Right - handed   Right-handed

Exercise 131 :

The linking number $\mathcal{L}$ is half the sum of the signs of all crossings where different components of the link cross each other.

The writhe $w(L)$ is the sum of the signs of all crossings.

So we can calculate it by $w(L) = 2 \cdot \mathcal{L} + \#(\text{self-crossings})$.

If L is a link with components K_i, then we can write the above formula as

$$w(L) = \sum_{i \neq j} \mathcal{L}(K_i, K_j) + \sum_i w(K_i).$$

Exercise 132 :

Since the linking number is only affected by crossings of different components, self-crossings do not matter. If we change a left-handed self-crossing to a right-handed, it does not change the linking number. The same for twists because there are no crossings of different components. If we change a left-handed crossing of two different components into a right-handed crossing, then the sum of signs increases by 2 and so the linking number increases by 1.

1. skein relation for the linking number  $= \begin{cases} 1, & \text{if different} \\ 0, & \text{if same} \end{cases}$

2. skein relation for the linking number 

3. skein relation for the linking number  $= 0$.

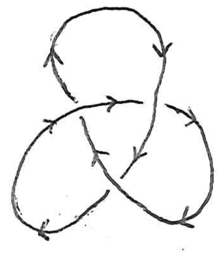
Exercise 133 :

The 'pancake-proof' is also know as the Seifert algorithm.

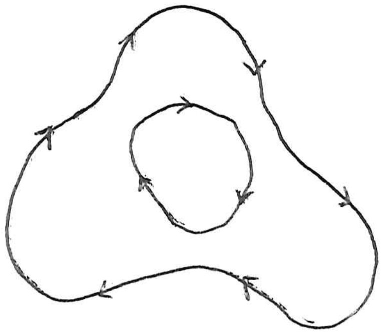
This algorithm shows that there exists a Seifert surface for every knot and that the orientation on the knot defines an orientation on the Seifert surface.

It is presented on the next page.

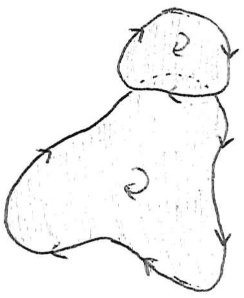
1. Given a knot K and a projection of that knot, introduce an orientation on K:



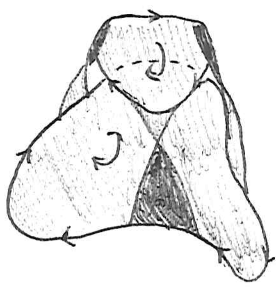
2. At each crossing of the projection, there will be two incoming and two outgoing strands. Connect each incoming strand to the adjacent outgoing strand, thus eliminating crossings, and creating a set of pancakes (Seifert circles) in the plane.



3. Move the circles out of the plane so they are at different heights and fill each one with a disk. The disks will have the same orientation as the knot.



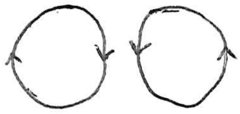
4. Finally, connect the disks with a series of twisted bands so that, when viewed from the top, the boundaries of the bands look like the original projection of the knot. The resulting surface has one boundary component, the knot K.

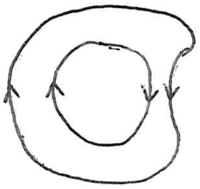


From here it follows automatically that the resulting surface is orientable because the twisting ramps just save the day.

In every case where two Seifert circles meet, the local appearance is as shown:

There are two ways to connect the ends of the lines, one which yields adjacent circles in the same plane, and one which yields concentric circles, which are moved into different planes by the algorithm:

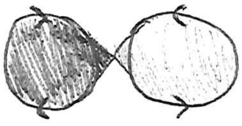


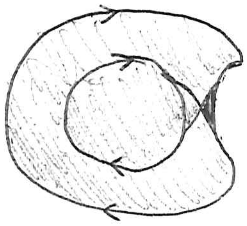


If the circles are adjacent, one has clockwise orientation and one has counterclockwise orientation. When we add the connecting band, the orientation is consistent.

If the circles are concentric, they both have the same orientation, and again when we add the connecting band, the orientation is consistent.

Therefore one can paint two sides of the surface with different colors, so it must be orientable.

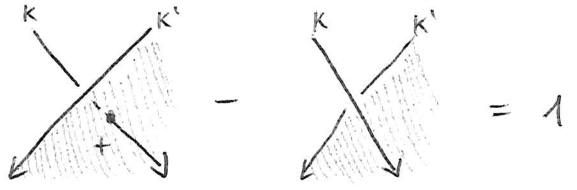


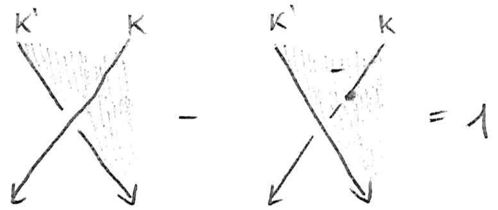


Exercise 134:

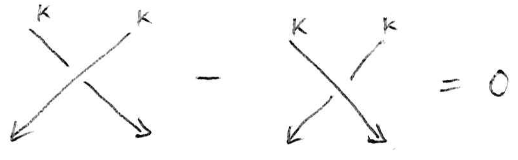
Skein relations for the intersection number:

different :



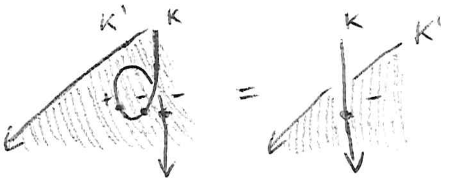


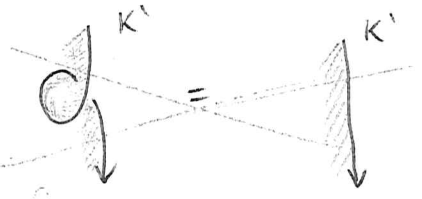
same :

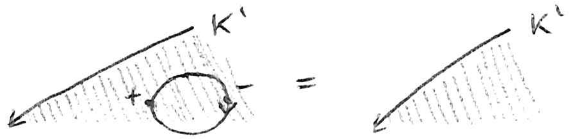


no change in intersection number

not possible because S oriented







not possible

↷ O = 0

Exercise 135 :

For the integral $w(K) = \int_{\mathbb{R}^3} A \wedge B$, we can choose any vector potential A with $dA = B$.

This means that we can add any 1-form C such that $dC = 0$. Let $A' = A + C$.

$$\Rightarrow \int_{\mathbb{R}^3} A' \wedge B = \int_{\mathbb{R}^3} A \wedge B + \int_{\mathbb{R}^3} C \wedge B$$
$$= w(K) + \int_{\mathbb{R}^3} C \wedge dA$$
$$= w(K) + \int_{\mathbb{R}^3} \underbrace{dC}_{=0} \wedge A$$
$$= w(K) .$$

Exercise 136 :

Let us write again $A_\alpha = (A_\alpha)_t \, dt + (A_\alpha)_r \, dr + (A_\alpha)_\theta \, d\theta$, then we have

$$\int_{\mathbb{R}^3} A_\alpha \wedge B_\beta = \int_{T_\beta} A_\alpha \wedge f_\beta \, r \, dr \wedge d\theta = \int_{T_\beta} (A_\alpha)_t \, dt \wedge f_\beta \, r \, dr \wedge d\theta$$
$$= \int_0^{2\pi} \int_0^1 \int_0^{2\pi} (A_\alpha)_t f_\beta \, r \, dt \, dr \, d\theta = \int_0^{2\pi} \int_0^1 f_\beta \, r \, dr \, d\theta \; \mathcal{L}(K_\alpha, K_\beta)$$
$$= \left(\int_{D^2} f_\alpha \, r \, dr \wedge d\theta \right) \left(\int_{D^2} f_\beta \, r \, dr \wedge d\theta \right) \mathcal{L}(K_\alpha, K_\beta)$$
$$= \left(\int_{D^2} f_\alpha \, r \, dr \wedge d\theta \right) \left(\int_{D^2} f_\beta \, r \, dr \wedge d\theta \right) w(K) .$$

since $\int_{D^2} f \, r \, dr \wedge d\theta =$

since $\mathcal{L}(K_\alpha, K_\beta) = w(K)$ for $\alpha \neq \beta$.

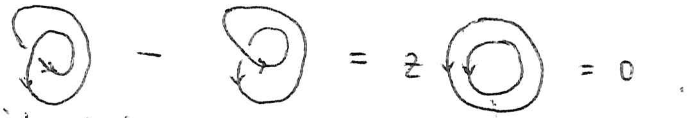
Exercise 137 :

We write $\nabla_L(z) = \sum_{i=0}^{\infty} a_i z^i = a_0 + a_1 z + a_2 z^2 + \dots$

According to the skein relations, there can only be a constant term a_0 if an unknot appears during the calculation. Like mentioned in the book, the Alexander polynomial for an unlinked unknot is zero, so if we have more then one component $a_0 = 0$. If we have exactly one component then only one unknot appears in the process which is equal to $a_0 = 1$.

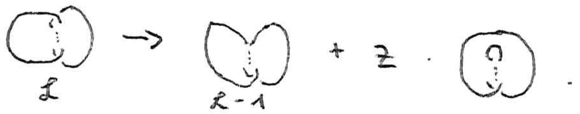
The linear term $a_1 z$ can only exist if we have exactly two components, because for one we always get an unlinked unknot,

or a Hopf link after the first usage of skein relation 1, which destroys the linear term.



If we have more then two components, the constant term vanisches and in the linear term appears a link with at least two components which gives another z and destroys linearity.

For two components, we get always a knot with two components and linking number $\mathcal{L}-1$ and a knot with one component when we use the first skein relation. We can imagine this as follows,



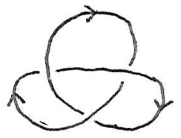
Since we're interested in the linear term, we use the fact that a link with one component gives $a_0 = 1$.

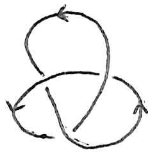
We continue this process till the link with two components has linking number zero, that is two unlinked unknots. And everytime, we get a z

$$\Rightarrow \nabla_L(z) = \nabla_{L_1}(z) + z \nabla_{L_0}(z) = \left(\nabla_{L_2}(z) + z \nabla_{L_0'}(z) \right) + z + \mathcal{O}(z^2) = \dots = \overset{\overset{0}{\shortparallel}}{\nabla_{oo}(z)} + \mathcal{L} z + \mathcal{O}(z^2) .$$

Exercise 138 :

Taking the mirror image L^* of an oriented link L corresponds to changing right-handed crossings to left-handed crossings and viseversa and changing the orientation :





and the actual handedness of the crossings doesn't change ↗

Since both, the handedness and the orientation changes simultaneous, the skein relations doesn't change too, since

 $= 1$.

So when we try to calculate the Alexander polynomial of L^*, we get the same result.

Exercise 139 :

There is no ambiguity about what to do, since the correct view is  and this is symmetric under rotations of 180°, such as the results of A and B.
So it doesn't matter how you rotate it, the result stays the same, no matter how you look at it.

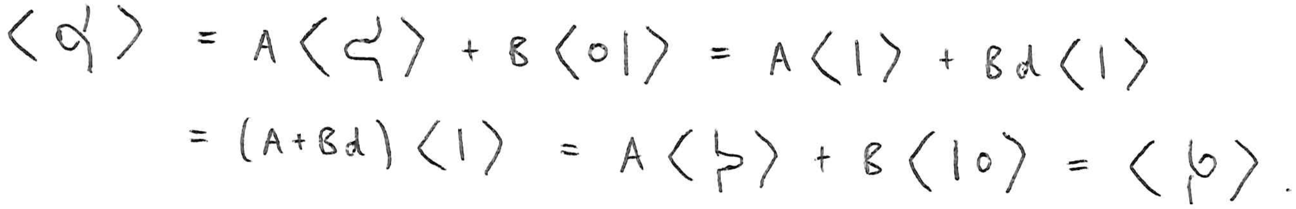
Exercise 140 :

$$\underset{\overset{\shortparallel}{E}}{-\frac{d}{d\beta}\ln Z(\beta)} = -\frac{1}{Z(\beta)}\frac{d}{d\beta}Z(\beta) = -\frac{1}{Z(\beta)}\sum_{\text{states } s}(-E(s))\,e^{-\beta E(s)} = \frac{1}{Z(\beta)}\sum_{\text{states } s}E(s)\,e^{-\beta E(s)}.$$

Exercise 141 :

Use that $\beta = \frac{1}{kT}$ $\Rightarrow$ $\frac{dE}{dT} = \frac{d\beta}{dT}\frac{d}{d\beta}\left(-\frac{d}{d\beta}\ln Z(\beta)\right) = k\beta^2\frac{d^2}{d\beta^2}\ln Z(\beta)$.

Exercise 142 :



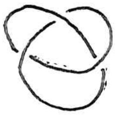
And $A + Bd = A - A^{-1}(A^2 + A^{-2}) = \cancel{A} - \cancel{A} - A^{-3} = -A^{-3}$.

Exercise 143 :

We use that the Kauffman bracket of the Hopf link is

 $= (A^2+B^2)d^2 + 2ABd = (A^2+A^{-2})^3 - 2(A^2+A^{-2})$.

Hopf link   two left-handed twist and an unknot

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$$= A\left[(A^2+A^{-2})^3 - 2(A^2+A^{-2})\right] - A^{-1}\left[A^{-6}(A^2+A^{-2})\right]$$

$$= A(A^2+A^{-2})^3 - 2A(A^2+A^{-2}) - A^{-7}(A^2+A^{-2})$$

$$= (A^2+A^{-2})\left[A(A^4+2+A^{-4}) - 2A - A^{-7}\right]$$

$$= -(A^2+A^{-2})(-A^5 - A^{-3} + A^{-7}).$$

The unknot can have right-handed and left-handed twists. To get rid of a right-handed twist we multiply by $-A^3$ and to get rid of a left-handed twist we multiply by $-A^{-3}$. In the end, we are left with an unknot which gives a factor $-(A^2+A^{-2})$.

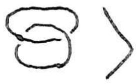
$$\Rightarrow \langle \text{unknot} \rangle = (-A^3)^{\#(\times)} (-A^{-3})^{\#(\times)} (-1)(A^2+A^{-2})$$

$$= (-A^3)^{\#(\times) - \#(\times)} (-1)(A^2+A^{-2})$$

$$= -(-A^3)^w (A^2+A^{-2}).$$

Since $\langle \text{trefoil} \rangle \neq \langle \text{unknot} \rangle$, they are not isotopic.

Exercise 144:

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$$= -A\left[A^6(A^2+A^{-2})\right] + A^{-1}\left[(A^2+A^{-2})^3 - 2(A^2+A^{-2})\right]$$

$$= -A^7(A^2+A^{-2}) + A^{-1}(A^2+A^{-2})^3 - 2A^{-1}(A^2+A^{-2})$$

$$= (A^2+A^{-2})\left[-A^7 + A^{-1}(A^4+2+A^{-4}) - 2A^{-1}\right]$$

$$= -(A^2+A^{-2})(A^7 - A^3 - A^{-5}).$$

Since the Kauffman brackets of the trefoil and its mirror image are not equal, they are not isotopic.

Exercise 145:

We have to show $\langle L^* \rangle(A) = \langle L \rangle(A^{-1})$ for any framed link L. Since $\langle L \rangle$ is calculated by the skein relations, we need to check that they obey this property:

$$\langle \times^* \rangle(A) = \langle \times \rangle(A) = A\langle \asymp \rangle + A^{-1}\langle || \rangle = A^{-1}\langle || \rangle + (A^{-1})^{-1}\langle \asymp \rangle = \langle \times \rangle(A^{-1})$$

$$\langle \alpha^* \rangle(A) = \langle \alpha \rangle(A) = -A^{-3}\langle | \rangle = -(A^{-1})^3\langle | \rangle = \langle \alpha \rangle(A^{-1})$$

$$\langle \alpha'^* \rangle(A) = \ldots = \langle \alpha' \rangle(A^{-1})$$

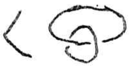
$$\langle \bigcirc^* \rangle(A) = \langle \bigcirc \rangle(A) = -(A^2+A^{-2}) = -((A^{-1})^2+(A^{-1})^{-2}) = -(A^{-2}+A^2) = \langle \bigcirc \rangle(A^{-1}).$$

Exercise 146 :

mirror image of trefoil

$\langle$  $\rangle$ $= A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

Hopf link

$= -A(A^2+A^{-2})(A^7-A^3-A^{-5}) - A^{-1}\cdot A^{-3} \langle$  $\rangle$

$$= -A(A^2+A^{-2})(A^7-A^3-A^{-5}) - A^{-4}\left((A^2+A^{-2})^3 - 2(A^2+A^{-2})\right)$$

$$= -(A^2+A^{-2})\left[A^8 - A^4 - A^{-4} + A^{-4}(A^4+2+A^{-4}) - 2A^{-4}\right]$$

$$= -(A^2+A^{-2})\left[A^8 - A^4 - A^{-4} + 1 + A^{-8}\right].$$

If we have $\langle K\rangle(A) = \langle K\rangle(A^{-1}) = \langle K^*\rangle(A)$, it means that the mirror image is isotopic to its normal image. In our case, we have

$$\langle K\rangle(A^{-1}) = -\left((A^{-1})^2 + (A^{-1})^{-2}\right)\left[(A^{-1})^8 - (A^{-1})^4 - (A^{-1})^{-4} + 1 + (A^{-1})^{-8}\right]$$

$$= -(A^{-2}+A^2)\left[A^{-8} - A^{-4} - A^4 + 1 + A^8\right]$$

$$= \langle K\rangle(A).$$

Exercise 147 :

Let's notice that

$$A^4 V_{\times}(A) = A^4 (-A^{-3})^{w(\times)} \langle \times \rangle(A) = -A\left(A\langle || \rangle(A) + A^{-1}\langle \asymp \rangle(A)\right)$$

$$= -A^2 \langle || \rangle(A) - \langle \asymp \rangle(A)$$

$$= -A^2 V_{\downarrow\downarrow}(A) - V_{\asymp}(A)$$

$$A^{-4} V_{\times}(A) = A^{-4} (-A^{-3})^{w(\times)} \langle \times \rangle(A) = -A^{-1}\left(A\langle \asymp \rangle(A) + A^{-1}\langle || \rangle(A)\right)$$

$$= -\langle \asymp \rangle(A) - A^{-2}\langle || \rangle(A)$$

$$= -V_{\asymp}(A) - A^{-2} V_{\downarrow\downarrow}(A).$$

$$\Rightarrow \quad A^4 V_{\times}(A) - A^{-4} V_{\times}(A) = (A^{-2} - A^2) V_{\downarrow\downarrow}(A).$$

Or $\quad q^{-1} \times - q \times = (q^{1/2} - q^{-1/2}) \downarrow\downarrow.$

For the last skein relation, we just notice that $\langle K \cup O\rangle = -(A^2+A^{-2})\langle K\rangle \ \forall K$, so every unknot makes a common prefactor and the possible twist are canceled out by the $(-A^{-3})^{w(K)}$ of the Jones Polynomial. So we can define $\langle O\rangle = 1$.

Moreover the Jones Polynomial is invariant under Reidemeister Move I.