Part III

Exercise 1 :

Let $X = v_1 \otimes \dots \otimes v_r \otimes \omega_1 \otimes \dots \otimes \omega_s$ and $Y = \omega_j(v_i)\, v_1 \otimes \dots \hat{v}_i \dots \otimes v_r \otimes \omega_1 \dots \hat{\omega}_j \dots \otimes \omega_s$.

We then have $X^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s} = X(dx^{\alpha_1}, \dots, dx^{\alpha_r}, \partial_{\beta_1}, \dots, \partial_{\beta_s}) = v_1^{\alpha_1} \dots v_r^{\alpha_r} \omega_{1\beta_1} \dots \omega_{s\beta_s}$

And
$$Y^{\alpha_1 \dots \hat{\alpha}_i \dots \alpha_r}_{\beta_1 \dots \hat{\beta}_j \dots \beta_s} = Y(dx^{\alpha_1}, \dots, \hat{dx}^{\alpha_i}, \dots, dx^{\alpha_r}, \partial_{\beta_1}, \dots, \hat{\partial}_{\beta_j}, \dots, \partial_{\beta_s}) = \omega_{j\mu} v_i^{\mu}\, v_1^{\alpha_1} \dots \hat{v}_i^{\alpha_i} \dots v_r^{\alpha_r} \omega_{1\beta_1} \dots \hat{\omega}_{j\beta_j} \dots \omega_{s\beta_s}$$
$$= v_1^{\alpha_1} \dots v_i^{\mu} \dots v_r^{\alpha_r} \omega_{1\beta_1} \dots \omega_{j\mu} \dots \omega_{s\beta_s}$$
$$= X^{\alpha_1 \dots \mu \dots \alpha_r}_{\beta_1 \dots \mu \dots \beta_s}.$$

Exercise 2 :

We show that D° is the Levi-Civita connection for any metric on $\mathbb{R}^n$ such that the components $g_{\alpha\beta}$ are constant with respect to the coordinate vector fields.

Write $g = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$. Choose $u = u^\alpha \partial_\alpha$, $v = v^\beta \partial_\beta$ and $w = w^\gamma \partial_\gamma$.

1. metric preserving :
$$u\, g(v,w) = u^\alpha \partial_\alpha (g_{\beta\gamma} v^\beta w^\gamma) = u^\alpha g_{\beta\gamma} (\partial_\alpha v^\beta) w^\gamma + u^\alpha g_{\beta\gamma} v^\beta (\partial_\alpha w^\gamma)$$
$$= g(D^\circ_u v, w) + g(v, D^\circ_u w).$$

2. torsion free :
$$D^\circ_v w - D^\circ_w v = v^\beta \partial_\beta (w^\gamma \partial_\gamma) - w^\gamma \partial_\gamma (v^\beta \partial_\beta)$$
$$= v^\beta (\partial_\beta w^\gamma) \partial_\gamma - w^\gamma (\partial_\gamma v^\beta) \partial_\beta$$
$$= [v^\beta \partial_\beta, w^\gamma \partial_\gamma]$$
$$= [v, w].$$

Exercise 3 :

We have
$$2 g_{\delta\gamma} \Gamma^\delta_{\alpha\beta} = \partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} \quad \Big|\, :2 \,\Big|\, \cdot g^{\gamma\lambda}$$
$$\Leftrightarrow \underbrace{g_{\delta\gamma} g^{\gamma\lambda}}_{=\delta^\lambda_\delta} \Gamma^\delta_{\alpha\beta} = \frac{1}{2} g^{\gamma\lambda} (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta})$$
$$\Leftrightarrow \Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\gamma\lambda} (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}).$$

Exercise 4 :

More generally we would have

I : $\partial_\alpha g_{\beta\gamma} = g(\nabla_\alpha e_\beta, e_\gamma) + g(e_\beta, \nabla_\alpha e_\gamma)$

II : $\partial_\beta g_{\gamma\alpha} = g(\nabla_\beta e_\gamma, e_\alpha) + g(e_\gamma, \nabla_\beta e_\alpha)$

III : $\partial_\gamma g_{\alpha\beta} = g(\nabla_\gamma e_\alpha, e_\beta) + g(e_\alpha, \nabla_\gamma e_\beta)$

With $\nabla_\alpha e_\beta - \nabla_\beta e_\alpha = [e_\alpha, e_\beta] = c^\gamma_{\alpha\beta} e_\gamma$

And $\nabla_\alpha e_\beta = \Gamma^\gamma_{\alpha\beta} e_\gamma$.

Taking again I + II - III one obtains :

$$\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} = g(\nabla_\alpha e_\beta, e_\gamma) + g(e_\beta, \nabla_\alpha e_\gamma) + g(\nabla_\beta e_\gamma, e_\alpha) + g(e_\gamma, \nabla_\alpha e_\beta - c^\gamma_{\alpha\beta} e_\gamma)$$
$$- g(\nabla_\alpha e_\gamma - c^\lambda_{\alpha\gamma} e_\lambda, e_\beta) - g(e_\alpha, \nabla_\beta e_\gamma - c^\lambda_{\beta\gamma} e_\lambda)$$
$$= 2\Gamma^\lambda_{\alpha\beta} g_{\lambda\gamma} - c^\lambda_{\alpha\beta} g_{\gamma\lambda} + c^\lambda_{\alpha\gamma} g_{\lambda\beta} + c^\lambda_{\beta\gamma} g_{\alpha\lambda}.$$

Exercise 5 :

In a basis of coordinate vector fields we have

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}\left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\beta\gamma}\right) = \frac{1}{2} g^{\alpha\delta}\left(\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\gamma\beta}\right) = \Gamma^{\alpha}_{\gamma\beta}.$$

While in an orthonormal basis, where $g_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $g_{\alpha\beta} = \pm 1$ if $\alpha = \beta$, we have

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(c_{\alpha\beta\gamma} + c_{\gamma\alpha\beta} - c_{\beta\gamma\alpha}\right) = -\frac{1}{2}\left(c_{\beta\gamma\alpha} - c_{\alpha\beta\gamma} - c_{\gamma\alpha\beta}\right) \underset{c_{\alpha\beta\gamma} = -c_{\alpha\gamma\beta}}{=} -\frac{1}{2}\left(c_{\gamma\beta\alpha} + c_{\alpha\gamma\beta} - c_{\beta\alpha\gamma}\right)$$

$$= -\Gamma_{\gamma\beta\alpha}.$$

Exercise 6 :

Let $g = d\phi^2 + \sin^2\phi \, d\theta^2$, so $g_{\alpha\beta} = \begin{pmatrix} g_{\phi\phi} & g_{\phi\theta} \\ g_{\theta\phi} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{pmatrix}$.

$$\Gamma^{\phi}_{\phi\phi} = \frac{1}{2} g^{\phi\lambda}\left(\partial_\phi g_{\phi\lambda} + \partial_\phi g_{\phi\lambda} - \partial_\lambda g_{\phi\phi}\right) = 0$$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{1}{2} g^{\phi\phi}\left(\partial_\phi g_{\theta\phi} + \partial_\theta g_{\phi\phi} - \partial_\phi g_{\phi\theta}\right) = 0$$

$$\Gamma^{\phi}_{\theta\theta} = \frac{1}{2} g^{\phi\phi}\left(\partial_\theta g_{\phi\theta} + \partial_\theta g_{\phi\theta} - \partial_\phi g_{\theta\theta}\right) = \frac{1}{2}\cdot(1)\cdot(-2\sin\phi\cos\phi) = -\sin\phi\cos\phi.$$

$$\Gamma^{\theta}_{\phi\phi} = 0$$

$$\Gamma^{\theta}_{\phi\theta} = \Gamma^{\theta}_{\theta\phi} = \frac{1}{2}\frac{1}{\sin^2\phi}(2\sin\phi\cos\phi) = \frac{\cos\phi}{\sin\phi}$$

$$\Gamma^{\theta}_{\theta\theta} = 0.$$

And for the metric $g = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2\left(d\phi^2 + \sin^2\phi \, d\theta^2\right)$ we obtain,

$\Gamma^{t}_{\mu\nu} = 0$, except $\Gamma^{t}_{rt} = \Gamma^{t}_{tr} = \frac{f'(r)}{f(r)}$.

$\Gamma^{r}_{tt} = f(r)^3 f'(r)$, $\Gamma^{r}_{t\mu} = \Gamma^{r}_{\mu t} = 0 \quad \forall \mu \neq t$

$\Gamma^{r}_{rr} = -\frac{f'(r)}{f(r)}$, $\Gamma^{r}_{r\mu} = \Gamma^{r}_{\mu r} = 0 \quad \forall \mu \neq r$

$\Gamma^{r}_{\phi\phi} = -r f(r)^2$, $\Gamma^{r}_{\phi\mu} = \Gamma^{r}_{\mu\phi} = 0 \quad \forall \mu \neq \phi$

$\Gamma^{r}_{\theta\theta} = -r\sin^2\phi \, f(r)^2$, $\Gamma^{r}_{\theta\mu} = \Gamma^{r}_{\mu\theta} = 0 \quad \forall \mu \neq \theta$.

$\Gamma^{\phi}_{\phi\phi} = \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = 0 = \Gamma^{\phi}_{t\mu} \quad \forall \mu$

$\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}$

$\Gamma^{\phi}_{r\theta} = \Gamma^{\phi}_{\theta r} = 0$

$\Gamma^{\phi}_{\theta\theta} = -\sin\phi\cos\phi$.

$\Gamma^{\theta}_{\phi\phi} = \Gamma^{\theta}_{\theta\theta} = 0 = \Gamma^{\theta}_{t\mu} \quad \forall \mu$

$\Gamma^{\theta}_{\phi\theta} = \Gamma^{\theta}_{\theta\phi} = \frac{\cos\phi}{\sin\phi}$

$\Gamma^{\theta}_{r\phi} = \Gamma^{\theta}_{\phi r} = 0$

$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = 1/r$.

Exercise 7 :

First, we want to show that if $(\nabla_\mu v)^\alpha = \partial_\mu v^\alpha + \Gamma^\alpha_{\mu\gamma} v^\gamma$ for a vector field v, then $(\nabla_\mu \omega)_\beta = \partial_\mu \omega_\beta - \Gamma^\lambda_{\mu\beta}\omega_\lambda$ for a covector field ω. Since $\omega(v) = \omega_\mu v^\mu$ is a scalar, its covariant derivative must be equal to the partial derivative $\nabla_\mu(\omega_\nu v^\nu) = \partial_\mu(\omega_\nu v^\nu)$.

So
$$\begin{aligned}\nabla_\mu(\omega_\nu v^\nu) &= (\nabla_\mu \omega_\nu) v^\nu + \omega_\nu (\nabla_\mu v^\nu)\\ &= (\partial_\mu \omega_\nu + \tilde{\Gamma}^\lambda_{\mu\nu}\omega_\lambda) v^\nu + \omega_\nu(\partial_\mu v^\nu + \Gamma^\nu_{\mu\gamma} v^\gamma)\\ &= (\partial_\mu \omega_\nu) v^\nu + \omega_\nu(\partial_\mu v^\nu) + \tilde{\Gamma}^\lambda_{\mu\nu}\omega_\lambda v^\nu + \Gamma^\nu_{\mu\gamma}\omega_\nu v^\gamma\end{aligned}$$

and $\partial_\mu(\omega_\nu v^\nu) = (\partial_\mu \omega_\nu) v^\nu + \omega_\nu(\partial_\mu v^\nu)$.

$$\Rightarrow \quad 0 = \tilde{\Gamma}^\lambda_{\mu\nu}\omega_\lambda v^\nu + \Gamma^\nu_{\mu\gamma}\omega_\nu v^\gamma = (\tilde{\Gamma}^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\nu})\omega_\lambda v^\nu$$

$$\Rightarrow \quad \tilde{\Gamma}^\lambda_{\mu\nu} = -\Gamma^\lambda_{\mu\nu} .$$

Now, remember Exercise 1, where we showed that $X^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_s} = V_1^{\alpha_1}\dots V_r^{\alpha_r}\omega_{1\beta_1}\dots\omega_{s\beta_s}$.

So
$$\begin{aligned}(\nabla_\mu X)^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_s} &= (\nabla_\mu V_1)^{\alpha_1} V_2^{\alpha_2}\dots\omega_{s\beta_s} + \dots + V_1^{\alpha_1}\dots(\nabla_\mu V_r)^{\alpha_r}\dots\omega_{s\beta_s} + V_1^{\alpha_1}\dots(\nabla_\mu\omega_1)_{\beta_1}\dots\omega_{s\beta_2} + \dots\\ &= (\partial_\mu V_1^{\alpha_1} + \Gamma^{\alpha_1}_{\mu\lambda}V_1^\lambda) V_2^{\alpha_2}\dots\omega_{s\beta_s} + \dots + V_1^{\alpha_1}\dots(\partial_\mu V_r^{\alpha_r} + \Gamma^{\alpha_r}_{\mu\lambda}V_r^\lambda)\dots\omega_{s\beta_s} + V_1^{\alpha_1}\dots(\partial_\mu\omega_{1\beta_1} - \Gamma^\lambda_{\mu\beta_1}\omega_\lambda)\\ &= (\partial_\mu V_1^{\alpha_1}) V_2^{\alpha_2}\dots\omega_{s\beta_s} + V_1^{\alpha_1}(\partial_\mu V_2^{\alpha_2})\dots\omega_{s\beta_s} + \dots + \Gamma^{\alpha_1}_{\mu\lambda}V_1^\lambda\dots\omega_{s\beta_s} + \Gamma^{\alpha_2}_{\mu\lambda}V_1^{\alpha_1}V_2^\lambda\dots\omega_{s\beta_s} + \dots - \Gamma^\lambda_{\mu\beta_1}\omega_\lambda V_1^{\alpha_1}\dots\\ &= \partial_\mu X^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_s} + \underbrace{\Gamma^{\alpha_1}_{\mu\lambda}X^{\lambda\dots\alpha_r}_{\beta_1\dots\beta_s} + \dots + \Gamma^{\alpha_r}_{\mu\lambda}X^{\alpha_1\dots\lambda}_{\beta_1\dots\beta_s}}_{r \text{ times}} \underbrace{- \Gamma^\lambda_{\mu\beta_1}X^{\alpha_1\dots\alpha_r}_{\lambda\dots\beta_s} - \dots - \Gamma^\lambda_{\mu\beta_s}X^{\alpha_1\dots\alpha_r}_{\beta_1\dots\lambda}}_{s \text{ times}} .\end{aligned}$$

Exercise 8 :

Choosing local coordinates and using $\nabla X = dx^\mu \otimes \nabla_\mu X$, one can easily show that

$\nabla(cX) = c\nabla X$, $\nabla(X + X') = \nabla X + \nabla X'$.

Using that $(X \otimes X')^{\alpha_1\dots\alpha_{r+r'}}_{\beta_1\dots\beta_{s+s'}} = X^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_s} X'^{\alpha_{r+1}\dots\alpha_{r+r'}}_{\beta_{s+1}\dots\beta_{s+s'}}$, we have

$$\begin{aligned}(\nabla_\mu(X\otimes X'))^{\alpha_1\dots\alpha_{r+r'}}_{\beta_1\dots\beta_{s+s'}} &= \nabla_\mu\left(X^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_s} X'^{\alpha_{r+1}\dots\alpha_{r+r'}}_{\beta_{s+1}\dots\beta_{s+s'}}\right)\\ &= \left(\nabla_\mu X^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_s}\right) X'^{\alpha_{r+1}\dots\alpha_{r+r'}}_{\beta_{s+1}\dots\beta_{s+s'}} + X^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_s}\left(\nabla_\mu X'^{\alpha_{r+1}\dots\alpha_{r+r'}}_{\beta_{s+1}\dots\beta_{s+s'}}\right) .\end{aligned}$$

Proofing that $Y^{\alpha_1\dots\hat{\alpha}_i\dots\alpha_r}_{\beta_1\dots\hat{\beta}_j\dots\beta_s} = X^{\alpha_1\dots\mu\dots\alpha_r}_{\beta_1\dots\mu\dots\beta_s} \Rightarrow \nabla_\rho Y^{\alpha_1\dots\hat{\alpha}_i\dots\alpha_r}_{\beta_1\dots\hat{\beta}_j\dots\beta_s} = \nabla_\rho X^{\alpha_1\dots\mu\dots\alpha_r}_{\beta_1\dots\mu\dots\beta_s}$ is just the same calculation like in Exercise 1.

Suppose we would have another $\tilde{\nabla}$ that satisfies all the above properties and $\tilde{\nabla} f = df$, then we would have $\tilde{\nabla} f = df = \nabla f \;\; \forall f \in C^\infty(M) \Rightarrow \nabla = \tilde{\nabla}$.

Exercise 9 :

The great circles are all covered by $\gamma : [0, 2\pi] \to \begin{pmatrix}\phi(t)\\ \theta(t)\end{pmatrix} = \begin{pmatrix} t \\ \theta_0\end{pmatrix}$ with $\theta_0 \in [0, 2\pi]$.

$$\left.\begin{aligned}\frac{d^2\gamma^\phi}{dt^2} = 0 \quad &+ \Gamma^\phi_{\theta\theta}\frac{d\gamma^\theta}{dt}\frac{d\gamma^\theta}{dt} = 0\\ \frac{d^2\gamma^\theta}{dt^2} = 0 \quad &+ 2\Gamma^\theta_{\phi\theta}\frac{d\gamma^\phi}{dt}\frac{d\gamma^\theta}{dt} = 0\end{aligned}\right\} \quad \frac{d^2\gamma^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda}\frac{d\gamma^\nu}{dt}\frac{d\gamma^\lambda}{dt} = 0 \quad \text{is fullfilled.}$$

Exercise 10 :

Since a path $\gamma(t)$ is a 1-dimensional manifold, its tangent vector $\gamma'(t)$ can be identified with the standard tangent vector $\frac{d}{dt}$ on 1-dimensional manifolds.
By chain rule, we obtain

$$\frac{d}{dt} g(v(t), w(t)) = \underbrace{\frac{dx^\mu(t)}{dt}}_{= \gamma'(t)} \partial_\mu g(v(t), w(t)) = g(\underbrace{\nabla_{\gamma'(t)} v(t)}_{=0}, w(t)) + g(v(t), \underbrace{\nabla_{\gamma'(t)} w(t)}_{=0}) = 0 .$$

Let's work out the Riemann tensor in a general basis e_α with $[e_\alpha, e_\beta] = c^\gamma_{\alpha\beta} e_\gamma$.

$$\Rightarrow R(e_\beta, e_\gamma) e_\delta = (\nabla_\beta \nabla_\gamma - \nabla_\gamma \nabla_\beta - \nabla_{[e_\beta, e_\gamma]}) e_\delta$$
$$= \nabla_\beta (\Gamma^\sigma_{\gamma\delta} e_\sigma) - \nabla_\gamma (\Gamma^\sigma_{\beta\delta} e_\sigma) - c^\lambda_{\beta\gamma} \nabla_{e_\lambda} e_\delta$$
$$= (\partial_\beta \Gamma^\sigma_{\gamma\delta}) e_\sigma + \Gamma^\sigma_{\gamma\delta} \Gamma^\tau_{\beta\sigma} e_\tau - (\partial_\gamma \Gamma^\sigma_{\beta\delta}) e_\sigma - \Gamma^\sigma_{\beta\delta} \Gamma^\tau_{\gamma\sigma} e_\tau - c^\lambda_{\beta\gamma} \Gamma^\sigma_{\lambda\delta} e_\sigma$$

$$\Rightarrow R^\alpha{}_{\beta\gamma\delta} = \partial_\beta \Gamma^\alpha_{\gamma\delta} - \partial_\gamma \Gamma^\alpha_{\beta\delta} - c^\lambda_{\beta\gamma} \Gamma^\alpha_{\lambda\delta} + \Gamma^\alpha_{\beta\sigma} \Gamma^\sigma_{\gamma\delta} - \Gamma^\alpha_{\gamma\sigma} \Gamma^\sigma_{\beta\delta} .$$

Exercise 11 :

For the standard metric on S^2 we obtain :

$$R^\phi{}_{\phi\theta\theta} = -R^\phi{}_{\theta\phi\theta} = \sin^2\phi$$

$$R^\theta{}_{\theta\phi\phi} = -R^\theta{}_{\phi\theta\phi} = 1$$

$$R_{\alpha\beta} = R^\phi{}_{\alpha\phi\beta} + R^\theta{}_{\alpha\theta\beta} = \begin{pmatrix} 0 & 0 \\ 0 & -\sin^2\phi \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -\sin^2\phi \end{pmatrix}$$

$$R = R_{\alpha\beta} g^{\alpha\beta} = tr\left(\begin{pmatrix} -1 & 0 \\ 0 & -\sin^2\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2}\phi \end{pmatrix} \right) = tr\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) = -2 .$$

For the spacetime metric $g = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2 (d\phi^2 + \sin^2\phi \, d\theta^2)$ we obtain :

$$R^t{}_{rtr} = \frac{f''(r) f(r) + f'(r)^2}{f(r)^2}$$

$$R^t{}_{\phi t\phi} = r f(r) f'(r)$$

$$R^t{}_{\theta t\theta} = r \sin^2\phi \, f(r) f'(r)$$

$$R^\phi{}_{t\phi t} = - \frac{f(r)^3 f'(r)}{r}$$

$$R^r{}_{\theta r\theta} = f(r) f'(r) r \sin^2\phi$$

$$R^r{}_{\phi r\phi} = r f(r) f'(r)$$

$$R^\phi{}_{\theta\phi\theta} = \sin^2\phi \, (f(r)^2 - 1)$$

$$R^\theta{}_{\phi\theta\phi} = f(r)^2 - 1$$

$$R^r{}_{trt} = - f(r)^2 f'(r)^2 - f(r)^3 f''(r)$$

$$R^\theta{}_{r\theta r} = \frac{f'(r)}{r f(r)}$$

$$R^\phi{}_{r\phi r} = \frac{f'(r)}{r f(r)}$$

$$R^\theta{}_{t\theta t} = - \frac{f(r)^3 f'(r)}{r}$$

$$R_{\alpha\beta} = R^t{}_{\alpha t\beta} + R^r{}_{\alpha r\beta} + R^\phi{}_{\alpha\phi\beta} + R^\theta{}_{\alpha\theta\beta}$$

$$= \begin{pmatrix} -f(r)^2 \left[f'(r)^2 + f(r) f''(r) + \frac{2 f(r) f'(r)}{r} \right] & 0 & 0 & 0 \\ 0 & \frac{1}{r} f(r)^{-2} \left[2 f'(r) f(r) + r f'(r)^2 + r f(r) f''(r) \right] & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\mathbb{0}_{2\times 2}} \\ 0 & 0 & & \end{pmatrix}$$

$$+ \begin{pmatrix} \multicolumn{2}{c}{\mathbb{0}_{2\times 2}} & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & -1 + f(r)^2 + 2r f(r) f'(r) & 0 \\ 0 & 0 & 0 & \sin^2\phi \left[2r f(r) f'(r) + f(r)^2 - 1 \right] \end{pmatrix}$$

$$R = R_{\alpha\beta} g^{\alpha\beta} = 2 \left(f'(r)^2 + f(r) f''(r) + \frac{f(r)^2}{r^2} \right) + \frac{8 f(r) f'(r)}{r} - \frac{2}{r^2}$$

Exercise 12 :

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\lambda} R^{\lambda}{}_{\beta\gamma\delta} = g(e_\alpha, e_\lambda) R^{\lambda}{}_{\beta\gamma\delta} = g(e_\alpha, R^{\lambda}{}_{\beta\gamma\delta}\, e_\lambda) = g(e_\alpha, R(e_\beta, e_\gamma) e_\delta) .$$

Exercise 13 :

$$\begin{aligned}
R^{\lambda}{}_{[\beta\gamma\delta]} &= \frac{1}{3!}\left(R^{\lambda}{}_{\beta\gamma\delta} - R^{\lambda}{}_{\gamma\beta\delta} + R^{\lambda}{}_{\gamma\delta\beta} - R^{\lambda}{}_{\delta\gamma\beta} + R^{\lambda}{}_{\delta\beta\gamma} - R^{\lambda}{}_{\beta\delta\gamma} \right) \\
&= \frac{1}{3!}\left(R^{\lambda}{}_{\beta\gamma\delta} + R^{\lambda}{}_{\beta\gamma\delta} + R^{\lambda}{}_{\gamma\delta\beta} + R^{\lambda}{}_{\gamma\delta\beta} + R^{\lambda}{}_{\delta\beta\gamma} + R^{\lambda}{}_{\delta\beta\gamma} \right) \\
&= \frac{1}{3}\left(R^{\lambda}{}_{\beta\gamma\delta} + R^{\lambda}{}_{\gamma\delta\beta} + R^{\lambda}{}_{\delta\beta\gamma} \right) \\
&= 0
\end{aligned}$$

$$\Leftrightarrow \quad R^{\lambda}{}_{\beta\gamma\delta} + R^{\lambda}{}_{\gamma\delta\beta} + R^{\lambda}{}_{\delta\beta\gamma} = 0 .$$

Exercise 14 :

Note that $R^{\lambda}{}_{\beta\gamma\delta} + R^{\lambda}{}_{\gamma\delta\beta} + R^{\lambda}{}_{\delta\beta\gamma} = 0 \quad \Leftrightarrow \quad R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0$.

And $R_{\alpha\delta\beta\gamma} + R_{\beta\alpha\delta\gamma} + R_{\delta\beta\alpha\gamma} = -R_{\gamma\delta\beta\alpha} - R_{\gamma\alpha\delta\beta} - R_{\gamma\beta\alpha\delta} = -(R_{\gamma\delta\beta\alpha} + R_{\gamma\beta\alpha\delta} + R_{\gamma\alpha\delta\beta}) = 0$.

So
$$\begin{aligned}
R_{\gamma\delta\alpha\beta} &= -R_{\beta\delta\alpha\gamma} \\
&= R_{\beta\alpha\gamma\delta} + R_{\beta\gamma\delta\alpha} \\
&= -R_{\delta\alpha\gamma\beta} - R_{\alpha\gamma\delta\beta} \\
&= R_{\delta\gamma\beta\alpha} + R_{\delta\beta\alpha\gamma} + R_{\alpha\delta\beta\gamma} + R_{\alpha\beta\gamma\delta} \\
&= 2R_{\alpha\beta\gamma\delta} + R_{\delta\beta\alpha\gamma} + R_{\alpha\delta\beta\gamma} \\
&= 2R_{\alpha\beta\gamma\delta} - R_{\beta\alpha\delta\gamma}
\end{aligned}$$

$$\Leftrightarrow \quad R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} .$$

$$\begin{aligned}
R_{[\alpha\beta\gamma\delta]} = \frac{1}{4!}\big(& R_{\alpha\beta\gamma\delta} - R_{\alpha\beta\delta\gamma} + R_{\alpha\delta\beta\gamma} - R_{\alpha\delta\gamma\beta} + R_{\alpha\gamma\delta\beta} - R_{\alpha\gamma\beta\delta} \\
& - R_{\beta\alpha\gamma\delta} + R_{\beta\alpha\delta\gamma} - R_{\beta\delta\alpha\gamma} + R_{\beta\delta\gamma\alpha} - R_{\beta\gamma\delta\alpha} + R_{\beta\gamma\alpha\delta} \\
& + R_{\gamma\alpha\beta\delta} - R_{\gamma\alpha\delta\beta} + R_{\gamma\beta\delta\alpha} - R_{\gamma\beta\alpha\delta} + R_{\gamma\delta\alpha\beta} - R_{\gamma\delta\beta\alpha} \\
& - R_{\delta\gamma\alpha\beta} + R_{\delta\gamma\beta\alpha} - R_{\delta\beta\gamma\alpha} + R_{\delta\beta\alpha\gamma} - R_{\delta\alpha\beta\gamma} + R_{\delta\alpha\gamma\beta} \big) \\
= \frac{1}{4!}\big(& R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} - (R_{\alpha\beta\delta\gamma} + R_{\alpha\delta\gamma\beta} + R_{\alpha\gamma\beta\delta}) \\
& + R_{\beta\alpha\delta\gamma} + R_{\beta\delta\gamma\alpha} + R_{\beta\gamma\alpha\delta} - (R_{\beta\alpha\gamma\delta} + R_{\beta\gamma\delta\alpha} + R_{\beta\delta\alpha\gamma}) \\
& + R_{\gamma\alpha\beta\delta} + R_{\gamma\beta\delta\alpha} + R_{\gamma\delta\alpha\beta} - (R_{\gamma\alpha\delta\beta} + R_{\gamma\delta\beta\alpha} + R_{\gamma\beta\alpha\delta}) \\
& + R_{\delta\gamma\beta\alpha} + R_{\delta\beta\alpha\gamma} + R_{\delta\alpha\gamma\beta} - (R_{\delta\gamma\alpha\beta} + R_{\delta\alpha\beta\gamma} + R_{\delta\beta\gamma\alpha}) \big) \\
= 0 & .
\end{aligned}$$

Exercise 15 :

$$\begin{aligned}
R^{\lambda}{}_{\alpha\lambda\beta} &= R_{\lambda\alpha}{}^{\lambda}{}_{\beta} = R_{\alpha}{}^{\lambda}{}_{\beta\lambda} = R_{\alpha\lambda\beta}{}^{\lambda} = R_{\alpha\beta} \\
R^{\lambda}{}_{\lambda\alpha\beta} &= R_{\lambda}{}^{\lambda}{}_{\alpha\beta} = R_{\alpha\beta}{}^{\lambda}{}_{\lambda} = R_{\alpha\beta\lambda}{}^{\lambda} = -R_{\alpha\beta} \\
R^{\lambda}{}_{\alpha\beta\lambda} &= R_{\lambda\alpha\beta}{}^{\lambda} = R_{\alpha\lambda}{}^{\lambda}{}_{\beta} = R_{\alpha}{}^{\lambda}{}_{\lambda\beta} = 0 .
\end{aligned}$$

Exercise 16 :

The tensor $R_{\alpha\beta} = \frac{1}{2} R g_{\alpha\beta}$ is symmetric and yields in 2 dimensions :

$R^\alpha_\alpha = \frac{1}{2} R \delta^\alpha_\alpha = R$, so it is the Ricci tensor.

For $R_{\alpha\beta\gamma\delta} = g_{\alpha\gamma} R_{\beta\delta} + g_{\beta\delta} R_{\alpha\gamma} - g_{\beta\gamma} R_{\alpha\delta} - g_{\alpha\delta} R_{\beta\gamma} - \frac{1}{2}(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) R$ we have

$$\begin{aligned} R^\lambda{}_{\beta\lambda\delta} &= \delta^\lambda_\lambda R_{\beta\delta} + g_{\beta\delta} R^\lambda_\lambda - g_{\beta\lambda} R^\lambda_\delta - \delta^\lambda_\delta R_{\beta\lambda} - \frac{1}{2}\left(\delta^\lambda_\lambda g_{\beta\delta} - \delta^\lambda_\delta g_{\beta\lambda}\right) R \\ &= 3 R_{\beta\delta} + g_{\beta\delta} R - R_{\beta\delta} - R_{\beta\delta} - \frac{1}{2}\left(3 g_{\beta\delta} - g_{\beta\delta}\right) R \\ &= R_{\beta\delta} . \end{aligned}$$

And it fulfills all the properties/symmetries of the Riemann tensor , so it is the Riemann tensor in 3 dimensions .

Exercise 17 :

Write $J = J_\alpha dx^\alpha$ for some 1-form on a Lorentzian manifold, signature $(n-1, 1)$.

$$\begin{aligned} \Rightarrow \quad * d * J &= * d\left(J_\alpha * dx^\alpha\right) \\ &= * d\left(J_\alpha \operatorname{sign}(i_1, \dots, i_n)\, \varepsilon(\alpha)\, dx^1 \wedge \dots \wedge \widehat{dx^\alpha} \wedge \dots \wedge dx^n\right) \\ &= * \left(\operatorname{sign}(i_1, \dots, i_n)\, \varepsilon(\alpha)\, \partial_\mu J_\alpha\, dx^\mu \wedge dx^1 \wedge \dots \wedge \widehat{dx^\alpha} \wedge \dots \wedge dx^n\right) \qquad \text{it has to be } \alpha = \mu \\ &= * \left(\operatorname{sign}(i_1, \dots, i_n)^2\, \varepsilon(\alpha)\, \partial_\alpha J_\alpha \underbrace{dx^1 \wedge \dots \wedge dx^\alpha \wedge \dots \wedge dx^n}_{= vol}\right) \\ &= \varepsilon(\alpha)\, \partial_\alpha J_\alpha \underbrace{* vol}_{= -1 \text{ for Lorentzian manifolds}} \\ &= - \varepsilon(\alpha)\, \partial_\alpha J_\alpha \end{aligned}$$

And now , since the Levi-Civita connection in local coordinates is just the standard flat connection , we have $\partial_\alpha = \nabla_\alpha$.

If we want to raise the index , the signature says us that we have to put a minus sign for one index but this is just cancelled out by the minus sign of $\varepsilon(\alpha)$.

$$\Rightarrow \quad * d * J = - \varepsilon(\alpha)\, \partial_\alpha J_\alpha = - \varepsilon(\alpha)\, \nabla_\alpha J_\alpha = - \nabla^\alpha J_\alpha .$$

Exercise 18 :

$$\begin{aligned} \nabla^\mu T_{\mu\nu} &= - \operatorname{tr}\left((\nabla^\mu F_{\mu\lambda}) F_\nu{}^\lambda + F_{\mu\lambda} (\nabla^\mu F_\nu{}^\lambda) - \frac{1}{2} g_{\mu\nu} (\nabla^\mu F^{\alpha\beta}) F_{\alpha\beta} \right) \\ &= - \operatorname{tr}\left((\nabla^\mu F_{\mu\lambda}) F_\nu{}^\lambda + F_{\mu\lambda} (\nabla^\mu F_\nu{}^\lambda) + \frac{1}{2} g_{\mu\nu} (\nabla^\alpha F^{\beta\mu} + \nabla^\beta F^{\mu\alpha}) F_{\alpha\beta} \right) \\ &= - \operatorname{tr}\left((\nabla^\mu F_{\mu\lambda}) F_\nu{}^\lambda + F_{\mu\lambda} (\nabla^\mu F_\nu{}^\lambda) + g_{\mu\nu} (\nabla^\alpha F^{\beta\mu}) F_{\alpha\beta} \right) \\ &= - \operatorname{tr}\left((\nabla^\mu F_{\mu\lambda}) F_\nu{}^\lambda + F_{\mu\lambda} (\nabla^\mu F_\nu{}^\lambda) - F_{\alpha\beta} (\nabla^\alpha F_\nu{}^\beta) \right) \\ &= - \operatorname{tr}\left((\nabla^\mu F_{\mu\lambda}) F_\nu{}^\lambda \right) \end{aligned}$$

To express the stress-energy tensor in terms of the electric and magnetic field, we remember that $-F \wedge *F = (\langle E,E\rangle - \langle B,B\rangle)\,\mathrm{vol}$ and that $\mathrm{tr}(F\wedge *F) = \frac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\,\mathrm{vol}$.

$$\Rightarrow \quad T_{\mu\nu} = -\mathrm{tr}\left(F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$
$$= -\mathrm{tr}\left(F_{i0}F_j{}^0 + F_{\mu i}F_\nu{}^i\right) - \frac{1}{2}g_{\mu\nu}\,\mathrm{tr}\left(\langle E,E\rangle - \langle B,B\rangle\right)$$
$$= -\mathrm{tr}\left(-\langle E_i,E_j\rangle + F_{\mu i}F_\nu{}^i\right) - \frac{1}{2}g_{\mu\nu}\,\mathrm{tr}\left(\langle E,E\rangle - \langle B,B\rangle\right)$$

$$F_{0i}F_0{}^i = -\langle E,E\rangle$$
$$F_{ki}F_0{}^i = -\varepsilon_{k\ell m}\langle E_\ell, B_m\rangle$$
$$F_{ji}F_k{}^i = -\langle B_j, B_k\rangle .$$

$$\Rightarrow \quad T_{\mu\nu} = \begin{pmatrix} T_{00} & \varepsilon_{1\ell m}\mathrm{tr}(\langle E_\ell,B_m\rangle) & \varepsilon_{2\ell m}\mathrm{tr}(\langle E_\ell,B_m\rangle) & \varepsilon_{3\ell m}\mathrm{tr}(\langle E_\ell,B_m\rangle) \\ \varepsilon_{1\ell m}\cdots & & & \\ \varepsilon_{2\ell m}\cdots & & \tau_{ij} = \mathrm{tr}\left(\langle E_i,E_j\rangle + \langle B_i,B_j\rangle - \frac{1}{2}g_{ij}(\langle E,E\rangle - \langle B,B\rangle)\right) & \\ \varepsilon_{3\ell m}\cdots & & & \end{pmatrix}.$$

$$T_{00} = \mathrm{tr}\left(\langle E,E\rangle - \frac{1}{2}\left(\langle E,E\rangle - \langle B,B\rangle\right)\right)$$
$$= \mathrm{tr}\left(\frac{1}{2}\left(\langle E,E\rangle + \langle B,B\rangle\right)\right).$$

This is the Yang-Mills equivalent of the electromagnetic energy density.

Exercise 19:

Regard R as an $\mathrm{End}(TM)$-valued 2-form, say $R = R_{\alpha\beta}\,dx^\alpha \wedge dx^\beta$.

$$\Rightarrow \quad d_\nabla R = [\nabla_\mu, R_{\alpha\beta}]\,dx^\mu \wedge dx^\alpha \wedge dx^\beta$$
$$= \frac{1}{3}\left([\nabla_\mu, R_{\alpha\beta}] + [\nabla_\alpha, R_{\beta\mu}] + [\nabla_\beta, R_{\mu\alpha}]\right)dx^\mu\wedge dx^\alpha \wedge dx^\beta$$
$$= 0.$$

If we use ∇_μ to define the exterior covariant derivative as $d_\nabla(\omega_I dx^I) = (\nabla_\mu \omega_I)\,dx^\mu \wedge dx^I$ we get for $d_\nabla R$ in local coordinates:

$$(d_\nabla R)^\lambda_\delta = (\nabla_\alpha R^\lambda{}_{\beta\gamma\delta})\,dx^\alpha\wedge dx^\beta \wedge dx^\gamma$$
$$= \frac{1}{3}\left(\nabla_\alpha R^\lambda{}_{\beta\gamma\delta} + \nabla_\beta R^\lambda{}_{\gamma\alpha\delta} + \nabla_\gamma R^\lambda{}_{\alpha\beta\delta}\right)dx^\alpha\wedge dx^\beta\wedge dx^\gamma \overset{!}{=} 0$$

$$\Rightarrow \quad \nabla_\alpha R^\lambda{}_{\beta\gamma\delta} + \nabla_\beta R^\lambda{}_{\gamma\alpha\delta} + \nabla_\gamma R^\lambda{}_{\alpha\beta\delta} = 0$$

$$\Rightarrow \quad \nabla_{[\alpha} R^\lambda{}_{\beta\gamma]\delta} = 0 .$$

Exercise 20:

Einstein's equation implies that $R_{\mu\nu} = 0$, because $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \Rightarrow R - \frac{n}{2} R = 0$ and for dimension $n=4$ this implies $R = 0$, so $R_{\mu\nu} = 0$.
From the results of Exercise 11, we see that this implies $R_{00}, R_{11}, R_{22}, R_{33} = 0$.
For $R_{22}, R_{33} = 0$, we get the condition

$$-1 + f(r)^2 + 2r f(r) f'(r) = 0$$

$$\Leftrightarrow \quad \frac{d}{dr}\left(r f(r)^2 \right) = 1 .$$

Differentiating this again with respect to r yields

$$4 f(r) f'(r) + 2r f(r)^2 + 2r f(r) f''(r) = 0$$

$$\Leftrightarrow \quad 2 f(r) f'(r) + r f(r)^2 + r f(r) f''(r) = 0 .$$

This is just $R_{00}, R_{11} = 0$. So all together $R_{\mu\nu} = 0$ really just implied $\frac{d}{dr}\left(r f(r)^2\right) = 1$.

Exercise 21:

Our symmetries of the Riemann tensor are equivalent to $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$ $\quad R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$
So we can consider R_{IJ} with $I = \{\alpha, \beta\}$, $J = \{\gamma, \delta\}$. $\quad R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$.
So it's like we have a skew-symmetric $n \times n$-matrix for I and J, and also a symmetric $D \times D$-matr
where D is the number of independent components of I and J.
A skew-symmetric matrix has $D = \frac{n(n-1)}{2}$ independent components.
A symmetric matrix has $\frac{D(D+1)}{2}$ independent components.
So till now, we get $\frac{n(n-1)}{4}\left(\frac{n(n-1)}{2} + 1\right)$ independent components.
But there is still the Bianchi identity, which implies $R_{[\alpha\beta\gamma\delta]} = 0$.
The number of equations of this type is equal to the number of ways one can choose 4 distinct indices from n. This is just $\binom{n}{4} = \frac{n!}{(n-4)!\,4!}$, so the final answer of independent components of the Riemann tensor in n dimensions is:

$$\frac{n(n-1)}{4}\left(\frac{n(n-1)}{2} + 1\right) - \frac{n!}{(n-4)!\,4!} = \frac{(n-4)!\,3!\,n(n-1)}{(n-4)!\,4!}\left(\frac{n(n-1)}{2} + 1\right) - \frac{n!}{(n-4)!\,4!}$$

$$= \frac{(n-4)!\,3!\,n^2(n-1)^2 + 2(n-4)!\,3!\,n(n-1) - 2n!}{2(n-4)!\,4!}$$

$$= \frac{n^2(n-1)^2}{8} + \frac{n(n-1)}{4} - \frac{n(n-1)(n-2)(n-3)}{4!}$$

$$= \frac{3n^4 - 6n^3 + 3n^2 + 6n^2 - 6n - n(n^2 - 3n + 2)(n-3)}{24}$$

$$= \frac{3n^4 \cancel{- 6n^3} \cancel{+ 9n^2} \cancel{- 6n} - n(n^3 \cancel{- 3n^2} \cancel{- 3n^2} \cancel{+ 9n} + 2n \cancel{- 6})}{24}$$

$$= \frac{2n^4 - 2n^2}{24} = \frac{n^2(n^2-1)}{12} .$$

Exercise 22

Consider the metric $g = L(u)^2 \left(e^{2\beta(u)} dx^2 + e^{-2\beta(u)} dy^2 \right) - du\,dv$.

The only non-zero Christoffel symbols are

$$\Gamma^x_{ux} = \frac{1}{2} g^{xx} (\partial_u g_{xx} + \partial_x g_{xu} - \partial_x g_{ux}) = \frac{1}{2} L(u)^{-2} e^{-2\beta(u)} \left(2L(u)L'(u)e^{2\beta(u)} + 2\beta'(u)L(u)^2 e^{2\beta(u)} \right) = \frac{L'(u)}{L(u)} + \beta'$$

$$\Gamma^y_{uy} = \frac{1}{2} g^{yy} (\partial_u g_{yy} + \partial_y g_{yu} - \partial_y g_{uy}) = \frac{1}{2} L(u)^{-2} e^{2\beta(u)} \left(2L(u)L'(u)e^{-2\beta(u)} - 2\beta'(u)L(u)^2 e^{-2\beta(u)} \right) = \frac{L'(u)}{L(u)} - \beta'$$

$$\Gamma^v_{xx} = -\frac{1}{2} g^{vu} \partial_u g_{xx} = L(u)L'(u)e^{2\beta(u)} + \beta'(u)L(u)^2 e^{2\beta(u)}$$

$$\Gamma^v_{yy} = -\frac{1}{2} g^{vu} \partial_u g_{yy} = L(u)L'(u)e^{-2\beta(u)} - \beta'(u)L(u)^2 e^{-2\beta(u)}.$$

For the Riemann tensor we have

$$R^x{}_{uxu} = \partial_u \Gamma^x_{xu} + \Gamma^x_{xu}\Gamma^x_{ux} = \frac{L''(u)}{L(u)} - \frac{L'(u)^2}{L(u)^2} + \beta''(u) + \left(\frac{L'(u)}{L(u)} + \beta'(u) \right)^2 = \frac{L''(u)}{L(u)} + \frac{2L'(u)\beta'(u)}{L(u)} + \beta'(u)^2 + \beta''(u)$$

$$R^y{}_{uyu} = \partial_u \Gamma^y_{yu} + \Gamma^y_{yu}\Gamma^y_{uy} = \frac{L''(u)}{L(u)} - \frac{L'(u)^2}{L(u)^2} - \beta''(u) + \left(\frac{L'(u)}{L(u)} - \beta'(u) \right)^2 = \frac{L''(u)}{L(u)} - \frac{2L'(u)\beta'(u)}{L(u)} + \beta'(u)^2 - \beta''(u)$$

And then we have also $R^v{}_{uxx}$ and $R^v{}_{uyy}$ but they don't enter into the Ricci tensor, so we don't calculate them.

$$\Rightarrow R_{\mu\nu} = R^x{}_{\mu x\nu} + R^y{}_{\mu y\nu} + \cancel{R^u{}_{\mu u\nu}} + \cancel{R^v{}_{\mu v\nu}}$$

The only non-vanishing component of the Ricci tensor is

$$R_{uu} = R^x{}_{uxu} + R^y{}_{uyu} = \frac{2L''(u)}{L(u)} + 2\beta'(u)^2.$$

Therefore the vaccum Einstein equations imply $R_{uu} = 0$, which is $L''(u) + \beta'(u)^2 L(u) = 0$.

When L is near to 1 and β is small, we get the linearized version of this equation:

$L(u) \approx 1$; $\quad L(u) \approx 1 + L'(u_0)(u - u_0)$

$$\Leftrightarrow \quad \beta'(u)^2 \left(1 + L'(u_0)(u-u_0) \right) = 0$$

$$\Rightarrow \quad \beta(u) = \text{const}.$$

$\beta(u) \approx 0$; $\quad \beta(u) \approx 0 + \beta'(u_0)(u-u_0)$

$$\Leftrightarrow \quad L''(u) + \beta'(u_0)^2 L(u) = 0$$

$$\Rightarrow \quad L(u) = A\cos(\beta'(u_0)u) + B\sin(\beta'(u_0)u).$$

All together these give ripples in the metric, which are equivalent to gravitational waves.

Exercise 23:

Suppose that A is diagonalizable, with eigenvalues $\lambda_1, \dots, \lambda_n$.
Then $1 + sA$ is also diagonalizable, with eigenvalues $1+s\lambda_1, \dots, 1+s\lambda_n$.
So we have

$$\begin{aligned} \det(1+sA) &= (1+s\lambda_1)\cdot \ldots \cdot (1+s\lambda_n) \\ &= 1 + s\cdot(\lambda_1 + \ldots + \lambda_n) + \ldots + s^n \lambda_1 \cdot \ldots \cdot \lambda_n \\ &= 1 + s\cdot \mathrm{tr}(A) + O(s^2). \end{aligned}$$

The diagonalizable matrices are dense in the space of all matrices, so this equation holds for all by continuity.

Exercise 24:

First note that
$$\begin{aligned}\nabla_\lambda \delta g_{\mu\nu} &= \partial_\lambda \delta g_{\mu\nu} - \Gamma^\rho_{\lambda\mu}\delta g_{\rho\nu} - \Gamma^\rho_{\lambda\nu}\delta g_{\mu\rho} \\ &= \delta\left(\partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho_{\lambda\nu} g_{\mu\rho}\right) + g_{\rho\nu}\delta\Gamma^\rho_{\lambda\mu} + g_{\mu\rho}\delta\Gamma^\rho_{\lambda\nu} \\ &= \delta\left(\nabla_\lambda g_{\mu\nu}\right) + g_{\rho\nu}\delta\Gamma^\rho_{\lambda\mu} + g_{\mu\rho}\delta\Gamma^\rho_{\lambda\nu} \\ &= g_{\rho\nu}\delta\Gamma^\rho_{\lambda\mu} + g_{\mu\rho}\delta\Gamma^\rho_{\lambda\nu}\,.\end{aligned}$$

Then calculate

$$\begin{aligned}\nabla_\mu \delta g_{\nu\lambda} + \nabla_\nu \delta g_{\lambda\mu} - \nabla_\lambda \delta g_{\mu\nu} &= g_{\rho\lambda}\delta\Gamma^\rho_{\mu\nu} + \cancel{g_{\nu\rho}\delta\Gamma^\rho_{\mu\lambda}} + \cancel{g_{\rho\mu}\delta\Gamma^\rho_{\nu\lambda}} + g_{\lambda\rho}\delta\Gamma^\rho_{\nu\mu} - \cancel{g_{\rho\nu}\delta\Gamma^\rho_{\lambda\mu}} - \cancel{g_{\mu\rho}\delta\Gamma^\rho_{\lambda\nu}} \\ &= 2 g_{\rho\lambda}\delta\Gamma^\rho_{\mu\nu}\,.\end{aligned}$$

$$\Rightarrow \quad \delta\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}\left(\nabla_\mu \delta g_{\nu\lambda} + \nabla_\nu \delta g_{\lambda\mu} - \nabla_\lambda \delta g_{\mu\nu}\right).$$

Exercise 25:

$$\begin{aligned}\delta R^\alpha{}_{\beta\gamma\eta} &= \delta\left(\partial_\beta \Gamma^\alpha_{\gamma\eta}\right) - \delta\left(\partial_\gamma \Gamma^\alpha_{\beta\eta}\right) + \delta\left(\Gamma^\sigma_{\gamma\eta}\Gamma^\alpha_{\beta\sigma}\right) - \delta\left(\Gamma^\sigma_{\beta\eta}\Gamma^\alpha_{\gamma\sigma}\right) \\ &= \partial_\beta \delta\Gamma^\alpha_{\gamma\eta} - \partial_\gamma \delta\Gamma^\alpha_{\beta\eta} + \Gamma^\alpha_{\beta\sigma}\delta\Gamma^\sigma_{\gamma\eta} + \Gamma^\sigma_{\gamma\eta}\delta\Gamma^\alpha_{\beta\sigma} - \Gamma^\alpha_{\gamma\sigma}\delta\Gamma^\sigma_{\beta\eta} - \Gamma^\sigma_{\beta\eta}\delta\Gamma^\alpha_{\gamma\sigma} + \Gamma^\sigma_{\beta\gamma}\delta\Gamma^\alpha_{\sigma\eta} - \Gamma^\sigma_{\beta\gamma}\delta\Gamma^\alpha_{\sigma\eta} \\ &= \partial_\beta \delta\Gamma^\alpha_{\gamma\eta} + \Gamma^\alpha_{\beta\sigma}\delta\Gamma^\sigma_{\gamma\eta} - \Gamma^\sigma_{\beta\gamma}\delta\Gamma^\alpha_{\sigma\eta} - \Gamma^\sigma_{\beta\eta}\delta\Gamma^\alpha_{\gamma\sigma} - \left(\partial_\gamma \delta\Gamma^\alpha_{\beta\eta} + \Gamma^\alpha_{\gamma\sigma}\delta\Gamma^\sigma_{\gamma\beta} - \Gamma^\sigma_{\gamma\beta}\delta\Gamma^\alpha_{\sigma\eta} - \Gamma^\sigma_{\gamma\eta}\delta\Gamma^\alpha_{\beta\sigma}\right) \\ &= \nabla_\beta \delta\Gamma^\alpha_{\gamma\eta} - \nabla_\gamma \delta\Gamma^\alpha_{\beta\eta}\,.\end{aligned}$$

Also, we can see that it is a special case of $\delta F = d_D \delta A$, where we have
$D \to \nabla$, $A^i_{\mu j} \to \Gamma^\alpha_{\mu\nu}$, $F^i_{\mu\nu j} \to R^\alpha_{\mu\nu\beta}$.
So in local coordinates,

$$\begin{aligned}\delta F &= \frac{1}{2}\delta R^\alpha{}_{\mu\nu\beta}\, dx^\mu \wedge dx^\nu = d_D \delta A = d_\nabla\left[\delta\Gamma^\alpha_{\nu\beta}\, dx^\nu\right] = \nabla_\mu \delta\Gamma^\alpha_{\nu\beta}\, dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}\left(\nabla_\mu \delta\Gamma^\alpha_{\nu\beta} - \nabla_\nu \delta\Gamma^\alpha_{\mu\beta}\right) dx^\mu \wedge dx^\nu\,.\end{aligned}$$

Exercise 26:

$$\begin{aligned}\delta R_{\alpha\beta} &= \nabla_\alpha \delta\Gamma^\gamma_{\gamma\beta} - \nabla_\gamma \delta\Gamma^\gamma_{\alpha\beta} \\ &= \frac{1}{2} g^{\gamma\lambda}\left(\cancel{\nabla_\alpha \nabla_\gamma \delta g_{\beta\lambda}} + \nabla_\alpha \nabla_\beta \delta g_{\gamma\lambda} - \cancel{\nabla_\alpha \nabla_\lambda \delta g_{\gamma\beta}}\right) - \frac{1}{2} g^{\gamma\lambda}\left(\nabla_\gamma \nabla_\alpha \delta g_{\beta\lambda} + \nabla_\gamma \nabla_\beta \delta g_{\alpha\lambda} - \nabla_\gamma \nabla_\lambda \delta g_{\alpha\beta}\right) \\ &= \frac{1}{2} g^{\gamma\lambda}\left(\nabla_\alpha \nabla_\beta \delta g_{\gamma\lambda} + \nabla_\gamma \nabla_\lambda \delta g_{\alpha\beta} - \nabla_\gamma\left(\nabla_\alpha \delta g_{\beta\lambda} + \nabla_\beta \delta g_{\alpha\lambda}\right)\right).\end{aligned}$$

Exercise 27:

$$\begin{aligned}\delta R &= \delta\left(g^{\alpha\beta} R_{\alpha\beta}\right) = \left(\delta g^{\alpha\beta}\right) R_{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta} \\ &= R_{\alpha\beta}\delta g^{\alpha\beta} + g^{\alpha\beta}\frac{1}{2} g^{\gamma\lambda}\left(\nabla_\alpha \nabla_\beta \delta g_{\gamma\lambda} + \nabla_\gamma \nabla_\lambda \delta g_{\alpha\beta} - \nabla_\gamma \nabla_\alpha \delta g_{\beta\lambda} - \nabla_\gamma \nabla_\beta \delta g_{\alpha\lambda}\right) \\ &= R_{\alpha\beta}\delta g^{\alpha\beta} + g^{\alpha\beta} g^{\gamma\lambda}\nabla_\alpha \nabla_\beta \delta g_{\gamma\lambda} - \frac{1}{2}\nabla^\lambda \nabla^\beta \delta g_{\beta\lambda} - \frac{1}{2}\nabla^\lambda \nabla^\alpha \delta g_{\alpha\lambda} \\ &= R_{\alpha\beta}\delta g^{\alpha\beta} + g^{\gamma\lambda}\nabla_\alpha \nabla^\alpha \delta g_{\gamma\lambda} - \nabla^\lambda \nabla^\alpha \delta g_{\alpha\lambda} \\ &= R_{\alpha\beta}\delta g^{\alpha\beta} + \nabla_\gamma \nabla^\gamma\left(g^{\alpha\beta}\delta g_{\alpha\beta}\right) - \nabla^\alpha \nabla^\beta \delta g_{\alpha\beta}\,.\end{aligned}$$

The linearized version of the vacuum Einstein equation ($R_{\mu\nu}=0$) where $g_{\mu\nu}$ is the Minkowski metric becomes:

$$0 = \partial_\mu \partial_\nu h^\alpha_\alpha + \Box h_{\mu\nu} - \partial^\alpha(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha}).$$

We make a plane wave ansatz:

$h_{\mu\nu} = h^{(0)}_{\mu\nu} e^{ik_\mu x^\mu}$.

In order for this to be a solution of the above equation, we have to have

$$0 = -k_\mu k_\nu \underbrace{h^{(0)\alpha}_{\alpha}}_{=0} - \underbrace{k^2}_{=0} h^{(0)}_{\mu\nu} + [k_\mu \underbrace{k^\alpha h^{(0)}_{\nu\alpha}}_{=0} + k_\nu \underbrace{k^\alpha h^{(0)}_{\mu\alpha}}_{=0}](-i)$$

traceless light-like gauge "perpendicular to wave vector"

A possible solution would be

$$h^{(0)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^{(0)}_{11} & h^{(0)}_{12} & 0 \\ 0 & h^{(0)}_{12} & -h^{(0)}_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

this is called the transverse tracelles gauge.

Exercise 29:

The vacuum Einstein equation is the same when deriving it from S_{EH} when the metric has arbitrary signature.

Now, consider $\mathcal{L} = R\,\mathrm{vol} + \frac{1}{2}\mathrm{tr}(F \wedge *F)$. We derive the equations of motion by varying the action $S = \int_M R\,\mathrm{vol} + \frac{1}{2}\mathrm{tr}(F\wedge *F)$ w.r.t. the metric and the Yang-Mills vector potential A.

Since the first term of the action doesn't depend on A, we get from the variation w.r.t. A just the normal Yang-Mills equation: $d_D * F = 0$.

For the other one, we obtain:

$$\delta S = \int_M \delta(R\,\mathrm{vol}) + \frac{1}{4}\delta[\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\,\mathrm{vol}]$$

$$= \int_M (R_{\alpha\beta} - \tfrac{1}{2}R g_{\alpha\beta})\,\mathrm{vol}\,\delta g^{\alpha\beta} + \frac{1}{2}\mathrm{tr}(g^{\mu\nu}F_{\mu\alpha}F_{\nu\beta})\,\mathrm{vol}\,\delta g^{\alpha\beta} + \frac{1}{4}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\,\delta\mathrm{vol}$$

$$= \int_M [R_{\alpha\beta} - \tfrac{1}{2}R g_{\alpha\beta} + \tfrac{1}{2}g^{\mu\nu}\mathrm{tr}(F_{\mu\alpha}F_{\nu\beta}) - \tfrac{1}{8}g_{\alpha\beta}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})]\,\mathrm{vol}\,\delta g^{\alpha\beta}$$

$$\rightsquigarrow \quad R_{\alpha\beta} - \tfrac{1}{2}R g_{\alpha\beta} + \tfrac{1}{2}\underbrace{\left(g^{\mu\nu}\mathrm{tr}(F_{\mu\alpha}F_{\nu\beta}) - \tfrac{1}{4}g_{\alpha\beta}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\right)}_{= T_{\alpha\beta}} = 0.$$

$= T_{\alpha\beta}$, Yang-Mills energy-momentum tensor.

Exercise 30:

There is a similar way to pullback $(0,s)$-tensors like differential forms with all the properties like linearity and naturality. In local coordinates we have $(\phi^* T)_{\mu_1\dots\mu_s} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_s}}{\partial x^{\mu_s}} T_{\alpha_1\dots\alpha_s}$.

Since the metric tensor is a $(0,2)$-tensor, we can pull it back.

Let $\phi: M \to M$ be a diffeomorphism. Then we have $\phi^*(dx'^\mu) = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$.

Since R is a scalar made up from $g_{\mu\nu}$ and ϕ^* is linear, we have also $\phi^* R(g) = R(\phi^* g)$ and so

$$S_{EH}(\phi^* g) = \int_M R(\phi^* g)\,\mathrm{vol}(\phi^* g) = \int_M \phi^* R(g) \cdot \left|\det\left(\tfrac{\partial x^\mu}{\partial x'^\nu}\right)\right| \sqrt{|\det(g'_{\mu\nu})|}\, d^n x'$$

$$= \int_M \phi^* R(g)\, \phi^*\sqrt{|\det(g_{\mu\nu})|}\, \phi^* d^n x = \int_M \phi^*(R(g)\,\mathrm{vol}(g)) = \int_M R(g)\,\mathrm{vol}(g) = S_{EH}(g).$$

xercise 31 :

To show : $g(e(s), e(s')) = \eta(s, s') \ \forall s, s' \in \Gamma(M\times\mathbb{R}^n) \Rightarrow g(e_I, e_J) = \eta_{IJ}$.

Write $s = s^I \xi_I$, $s' = s'^J \xi_J$ then we have

$$g(e(s), e(s')) = \eta(s, s') \qquad \forall s, s' \in \Gamma(M\times\mathbb{R}^n)$$
$$\Leftrightarrow \quad s^I s'^J g(e_I, e_J) = s^I s'^J \eta_{IJ} \qquad \forall s^I, s'^J \in C^\infty(M)$$
$$\Leftrightarrow \quad g(e_I, e_J) = \eta_{IJ}.$$

xercise 32 :

Since η_{IJ} is copied after the Minkowski metric, we can use it and its inverse η^{IJ} to raise and lower indices as we did before.

$$\Rightarrow \delta^I{}_J = \eta^{IK}\eta_{KJ} = \eta^I{}_J = g(e^I, e_J) = g_{\alpha\beta}\, e^{I,\alpha} e^\beta_J = e^I_\alpha e^\alpha_J .$$

Moreover, one can show that $\delta^\alpha_\beta = e^\alpha_I e^I_\beta$.

xercise 33 :

D is a Lorentz connection on $M\times\mathbb{R}^n \Leftrightarrow A^{IJ}_\mu = -A^{JI}_\mu$.

"$\Leftarrow$" obvious.

"$\Rightarrow$" Analyze the action of both sides. We have

$$v\eta(s,s') = v(s^I s'^J \eta_{IJ}) = v(s^I) s'^J \eta_{IJ} + s^I v(s'^J)\eta_{IJ}.$$
$$\eta(D_v s, s') + \eta(s, D_v s') = v(s^I)s'^J\eta_{IJ} + A^I_{\mu K} v^\mu s^K s'^J \eta_{IJ} + s^I v(s'^J)\eta_{IJ} + s^I A^J_{\mu K} v^\mu s'^K \eta_{IJ}$$
$$= v\eta(s,s') + (A^I_{\mu K} s^K s'^J + A^J_{\mu K} s'^K s^I) v^\mu \eta_{IJ}.$$

In order to obtain the condition for a Lorentz connection, it has to be

$$A_{\mu, JK} s^K s'^J + A_{\mu, IK} s'^K s^I = 0 \qquad \forall s^I, s'^J \in C^\infty(M).$$
$$\underset{\text{dummy indices}}{\Leftrightarrow} \quad (A_{\mu,JI} + A_{\mu,IJ}) s^I s'^J = 0 \qquad \forall s^I, s'^J \in C^\infty(M).$$
$$\Leftrightarrow \quad A_{\mu,IJ} = -A_{\mu,JI}.$$

By raising both indices with η_{IJ}, we also obtain $A^{IJ}_\mu = -A^{JI}_\mu$.

Exercise 34 :

If A is a Lorentz connection, then $F^{IJ}_{\alpha\beta} = -F^{IJ}_{\beta\alpha} = -F^{JI}_{\alpha\beta}$.

The antisymmetry of the greek indices comes from the property of the commutator and $F^{IJ}_{\alpha\beta} = \partial_\alpha A^{IJ}_\beta - \partial_\beta A^{IJ}_\alpha + [A_\alpha, A_\beta]^{IJ}$.

For the antisymmetry of the latin indices, we have to show that $[A_\alpha, A_\beta]^{IJ} = -[A_\alpha, A_\beta]^{JI}$.

This can be easily seen : $[A_\alpha, A_\beta]^{IJ} = A^I_{\alpha K} A^{KJ}_\beta - A^I_{\beta K} A^{KJ}_\alpha$

$$= -A^I_{\alpha K} A^{JK}_\beta + A^I_{\beta K} A^{JK}_\alpha$$
$$= -A^{IK}_\alpha A^J_{\beta K} + A^{IK}_\beta A^J_{\alpha K}$$
$$= -A^J_{\alpha K} A^{KI}_\beta + A^J_{\beta K} A^{KI}_\alpha$$
$$= -[A_\alpha, A_\beta]^{JI}.$$

Together with Ex. 33, we obtain $F^{IJ}_{\alpha\beta} = -F^{JI}_{\alpha\beta}$.

Exercise 35 :

We have to show that $\tilde{R}(\partial_\mu,\partial_\nu) = [\tilde{\nabla}_\mu,\tilde{\nabla}_\nu] - \tilde{\nabla}_{[\partial_\mu,\partial_\nu]} = [\tilde{\nabla}_\mu,\tilde{\nabla}_\nu]$.

Look at the definition of $\tilde{R}$, we obtain:

$$\tilde{R}^\gamma{}_{\alpha\beta\delta}\,\partial_\gamma = F^{IJ}_{\alpha\beta}\, e_{I,\delta}\, e^\gamma_J\, \partial_\gamma = \partial_\alpha A^{IJ}_\beta\, e_{I,\delta}\, e^\gamma_J\, \partial_\gamma - \partial_\beta A^{IJ}_\alpha\, e_{I,\delta}\, e^\gamma_J\, \partial_\gamma + A^I_{\alpha K} A^{KJ}_\beta\, e_{I,\delta}\, e^\gamma_J\, \partial_\gamma - A^I_{\beta K} A^{KJ}_\alpha\, e_{I,\delta}\, e^\gamma_J\, \partial_\gamma$$

$$\| \qquad \tilde{R}_{\alpha\beta}\,\partial_\delta \qquad \| \qquad \tilde{R}(\partial_\alpha,\partial_\beta)\,\partial_\delta$$

$$\overset{(*)}{=} -\left(\partial_\alpha \tilde{\Gamma}^\gamma_{\beta\delta}\right)\partial_\gamma + \left(\partial_\beta \tilde{\Gamma}^\gamma_{\alpha\delta}\right)\partial_\gamma - \tilde{\Gamma}^\lambda_{\alpha\gamma}\tilde{\Gamma}^\gamma_{\beta\delta}\,\partial_\lambda + \tilde{\Gamma}^\lambda_{\beta\gamma}\tilde{\Gamma}^\gamma_{\alpha\delta}\,\partial_\lambda$$

$$= -\tilde{\nabla}_\alpha\left(\tilde{\Gamma}^\gamma_{\beta\delta}\,\partial_\gamma\right) + \tilde{\nabla}_\beta\left(\tilde{\Gamma}^\gamma_{\alpha\delta}\,\partial_\gamma\right)$$

$$= -\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\,\partial_\delta + \tilde{\nabla}_\beta\tilde{\nabla}_\alpha\,\partial_\delta$$

$$= -[\tilde{\nabla}_\alpha,\tilde{\nabla}_\beta]\,\partial_\delta .$$

$\rightsquigarrow$ so maybe the definition in the book was wrong
With $\tilde{R}^\gamma{}_{\alpha\beta}{}^\delta = F^{IJ}_{\alpha\beta}\, e^\gamma_I\, e^\delta_J$ it works.

(*): $$\tilde{\Gamma}^\lambda_{\alpha\gamma}\tilde{\Gamma}^\gamma_{\beta\delta}\,\partial_\lambda = A^I_{\alpha J}\, e^J_\gamma\, e^\lambda_I\, A^K_{\beta L}\, e^L_\delta\, e^\gamma_K\, \partial_\lambda$$
$$= \delta^J{}_K\, A^I_{\alpha J} A^K_{\beta L}\, e^\lambda_I\, e^L_\delta\, \partial_\lambda$$
$$= A^I_{\alpha K} A^K_{\beta L}\, e^\lambda_I\, e^L_\delta\, \partial_\lambda$$
$$= A^I_{\alpha K} A^{KJ}_\beta\, e_{J,\delta}\, e^\gamma_I\, \partial_\gamma .$$
$$= A^I_{\beta K} A^{KJ}_\alpha\, e_{I,\delta}\, e^\gamma_J\, \partial_\gamma .$$

Exercise 36 :

$$\delta\mathrm{vol} = -\eta^{IJ} g_{\alpha\beta}\, e^\beta_J\, (\delta e^\alpha_I)\,\mathrm{vol}$$
$$= -\eta^{IJ}\,\eta_{KL}\, e^K_\alpha\, e^L_\beta\, e^\beta_J\, (\delta e^\alpha_I)\,\mathrm{vol}$$
$$= -\eta^{IJ}\,\eta_{KL}\,\delta^L_J\, e^K_\alpha\, (\delta e^\alpha_I)\,\mathrm{vol}$$
$$= -\eta^{IJ}\,\eta_{KJ}\, e^K_\alpha\, (\delta e^\alpha_I)\,\mathrm{vol}$$
$$= -\delta^I{}_K\, e^K_\alpha\, (\delta e^\alpha_I)\,\mathrm{vol}$$
$$= -e^K_\alpha\, (\delta e^\alpha_K)\,\mathrm{vol} .$$

Exercise 37 :

$$\delta S = 2\int_M \left(e^\beta_J F^{IJ}_{\alpha\beta} - \tfrac{1}{2}\, e^I_\alpha\, e^\gamma_K\, e^\delta_L\, F^{KL}_{\gamma\delta}\right)(\delta e^\alpha_I)\,\mathrm{vol}$$

$$\tilde{R}_{\alpha\gamma} = F^{IJ}_{\alpha\beta}\, e_{I,\gamma}\, e^\beta_J$$
$$\tilde{R} = F^{IJ}_{\alpha\beta}\, e^\alpha_I\, e^\beta_J$$

$$= 2\int_M \left(e^{I,\gamma}\,\tilde{R}_{\alpha\gamma} - \tfrac{1}{2}\, e^I_\alpha\,\tilde{R}\right)(\delta e^\alpha_I)\,\mathrm{vol}$$
$$= 2\int_M \left(g^{\gamma\delta}\, e^I_\delta\,\tilde{R}_{\alpha\gamma} - \tfrac{1}{2}\, e^I_\alpha\,\tilde{R}\right)(\delta e^\alpha_I)\,\mathrm{vol}$$
$$= 2\int_M g^{\gamma\delta}\, e^I_\delta \left(\tilde{R}_{\alpha\gamma} - \tfrac{1}{2}\, g_{\gamma\delta}\, e^\delta_I\, e^I_\alpha\,\tilde{R}\right)(\delta e^\alpha_I)\,\mathrm{vol}$$
$$= 2\int_M \eta^{KL}\, e^\gamma_K\, e^\delta_L\, e^I_\delta \left(\tilde{R}_{\alpha\gamma} - \tfrac{1}{2}\, g_{\alpha\gamma}\,\tilde{R}\right)(\delta e^\alpha_I)\,\mathrm{vol}$$
$$= 2\int_M \eta^{KI}\, e^\gamma_K \left(\tilde{R}_{\alpha\gamma} - \tfrac{1}{2}\, g_{\alpha\gamma}\,\tilde{R}\right)(\delta e^\alpha_I)\,\mathrm{vol}$$
$$= 2\int_M \left(\tilde{R}_{\alpha\beta} - \tfrac{1}{2}\, g_{\alpha\beta}\,\tilde{R}\right)\eta^{IJ}\, e^\beta_J\, (\delta e^\alpha_I)\,\mathrm{vol} .$$

Exercise 38:

$$\Rightarrow \delta\tilde{R} = \frac{1}{2} g^{\alpha\beta}\delta\tilde{R}_{\alpha\beta} = g^{\alpha\beta}\tilde{\nabla}_{[\alpha}\delta C^{\gamma}_{\gamma]\beta} = g^{\alpha\beta}\frac{1}{2}\left[\tilde{\nabla}_\alpha \delta C^\gamma_{\gamma\beta} - \tilde{\nabla}_\gamma\delta C^\gamma_{\alpha\beta}\right]$$

$$= g^{\alpha\beta}\frac{1}{2}\left[\nabla_\alpha\delta C^\gamma_{\gamma\beta} - C^\eta_{\alpha\beta}\delta C^\gamma_{\gamma\eta} - \nabla_\gamma\delta C^\gamma_{\alpha\beta} - C^\gamma_{\gamma\eta}\delta C^\eta_{\alpha\beta} + C^\eta_{\gamma\alpha}\delta C^\gamma_{\eta\beta} + C^\eta_{\gamma\beta}\delta C^\gamma_{\alpha\eta}\right]$$

$$= g^{\alpha\beta}\nabla_{[\alpha}\delta C^\gamma_{\gamma]\beta} + \frac{1}{2}g^{\alpha\beta}\left[-C^\gamma_{\gamma\eta}\delta C^\eta_{\alpha\beta} + C^\eta_{\gamma\alpha}\delta C^\gamma_{\eta\beta} - C^\eta_{\alpha\beta}\delta C^\gamma_{\gamma\eta} + C^\eta_{\gamma\beta}\delta C^\gamma_{\alpha\eta}\right].$$

Exercise 39:

Show: $g^{\alpha\beta}\left[-C^\gamma_{\gamma\eta}\delta C^\eta_{\alpha\beta} + C^\eta_{\gamma\alpha}\delta C^\gamma_{\eta\beta} - C^\eta_{\alpha\beta}\delta C^\gamma_{\gamma\eta} + C^\eta_{\gamma\beta}\delta C^\gamma_{\alpha\eta}\right] = 0 \iff C^\gamma_{\alpha\beta} = 0$.

"$\Leftarrow$": obvious.

"$\Rightarrow$":

$$0 = -C^\gamma_{\gamma\eta}\delta C^{\eta\alpha}_{\alpha} + C^\eta_{\gamma\alpha}\delta C^{\gamma\alpha}_{\eta} - C^{\eta\alpha}_{\alpha}\delta C^\gamma_{\gamma\eta} + C^{\eta\alpha}_{\gamma}\delta C^\gamma_{\alpha\eta}$$

$$= -\delta(C^\gamma_{\gamma\eta}C^{\eta\alpha}_{\alpha}) + (\delta C^{\gamma\alpha}_{\eta})C^\eta_{\gamma\alpha} + C^{\gamma\alpha}_{\eta}\delta C^\eta_{\alpha\gamma}$$

$$= \delta\left(C^{\gamma\alpha}_{\eta}C^\eta_{\gamma\alpha} - C^\gamma_{\gamma\eta}C^{\eta\alpha}_{\alpha}\right) + 2C^{\gamma\alpha}_{\eta}\delta C^\eta_{[\alpha\gamma]}$$

Now, more work should be done but we'll just say that these two variations are linearly independent and each of them has to be zero to fullfill the above equation.
Since $\delta C^\eta_{[\alpha\gamma]}$ is an arbitrary variation, we obtain $C^{\gamma\alpha}_{\eta} = 0$. With this, the first variation is automatically zero. Since $C^\sigma_{\mu\nu} = g^{\sigma\eta}g_{\mu\gamma}g_{\nu\alpha}C^{\gamma\alpha}_{\eta} \Rightarrow C^\sigma_{\mu\nu} = 0$.

Exercise 40:

Let D be a Lorentz connection and $\tilde{\nabla}$ the corresponding imitation Levi-Civita connection.
Define
$$s = e^{-1}v = e^I_\alpha v^\alpha \xi_I$$
$$s' = e^{-1}w = e^J_\beta w^\beta \xi_J$$

$$\Rightarrow \quad u\,\eta(s,s') = \eta(D_u s, s') + \eta(s, D_u s')$$

$$\Rightarrow u\,\eta(e^{-1}v, e^{-1}w) = \eta\left((u(s^J) + A^J_{\mu I}u^\mu s^I)\xi_J, s'^I\xi_I\right) + \eta\left(s^I\xi_I, (u(s'^J) + A^J_{\mu I}u^\mu s'^I)\xi_J\right)$$

$$\Rightarrow u\big(v^\alpha w^\beta \underbrace{e^I_\alpha e^J_\beta \eta_{IJ}}_{= g_{\alpha\beta}}\big) = \big(u(v^\alpha) + \underbrace{A^J_{\mu I'}u^\mu v^\gamma e^{I'}_\gamma e^\alpha_J}_{= \tilde{\Gamma}^\alpha_{\mu\gamma}v^\gamma u^\mu}\big)w^\beta \underbrace{e^J_\alpha e^I_\beta \eta_{IJ}}_{= g_{\alpha\beta}} + \big(u(w^\alpha) + \underbrace{A^J_{\mu I'}u^\mu w^\gamma e^{I'}_\gamma e^\alpha_J}_{= \tilde{\Gamma}^\alpha_{\mu\gamma}w^\gamma u^\mu}\big)v^\beta \underbrace{e^I_\beta e^J_\alpha \eta_{IJ}}_{= g_{\alpha\beta}}$$

$$\Rightarrow \quad u\,g(v,w) = g(\tilde{\nabla}_u v, w) + g(v, \tilde{\nabla}_u w).$$

So $\tilde{\nabla}$ is metric preserving. Since the only connection which is metric preserving and torsion free, is the Levi-Civita connection ∇, $\tilde{\nabla} = \nabla$ if and only if $\tilde{\nabla}$ is torsion free.

Exercise 41:

We have $e^{-1}\partial_\alpha = e^I_\alpha \xi_I$, so the inverse frame field $e^{-1}: TM \to M \times \mathbb{R}^n$ can be thought of as an $\mathbb{R}^n$-valued 1-form: $e^{-1} = e^I_\alpha \xi_I \otimes dx^\alpha$.
We have the correspondance $s = e^{-1}v$ or $s_\mu = e^I_\mu \xi_I$. So consider

$$d_D e^{-1} = D_\mu(e^I_\alpha \xi_I) \otimes dx^\mu \wedge dx^\alpha$$

$$= D_\mu(e^{-1}\partial_\alpha) \otimes dx^\mu \wedge dx^\alpha$$

since $\tilde{\nabla}$ is the corresponding to D in TM

$$\cong \tilde{\nabla}_\mu \partial_\alpha \otimes dx^\mu \wedge dx^\alpha$$

$$= \underbrace{\frac{1}{2}\left[\tilde{\nabla}_\mu\partial_\alpha - \tilde{\nabla}_\alpha\partial_\mu\right]}_{= (*)} \otimes dx^\mu \wedge dx^\alpha$$

Since we work in local coordinates, the connection $\tilde{\nabla}$ is only torsion free if
$$\tilde{\nabla}_\alpha\partial_\beta - \tilde{\nabla}_\beta\partial_\alpha = 0.$$
This is only the case, iff $(*) = 0$
$$\Rightarrow d_D e^{-1} = 0.$$

Exercise 42:

$$S(A,e) = \int_M e^\alpha_I e^\beta_J F^{IJ}_{\alpha\beta} \, \mathrm{vol} = \int_M e^{-1}_I \wedge e^{-1}_J \wedge * F^{IJ} = \int_M \mathrm{tr}\left(e^{-1} \wedge e^{-1} \wedge * F \right).$$

Varying with respect to A yields:

$$\delta S = \int_M \mathrm{tr}\left(e^{-1} \wedge e^{-1} \wedge * \delta F \right) = \int_M \mathrm{tr}\left(e^{-1} \wedge e^{-1} \wedge * d_D \delta A \right) = \pm \int_M \mathrm{tr}\left(d_D * (e^{-1} \wedge e^{-1}) \wedge \delta A \right)$$

$$= \pm \int_M \varepsilon^{ABCD} \, \mathrm{tr}\left(d_D (e^{-1}_C \wedge e^{-1}_D) \wedge \delta A \right) = \pm \int_M \varepsilon^{ABCD} \, \mathrm{tr}\left([d_D e^{-1}_C \wedge e^{-1}_D - e^{-1}_C \wedge d_D e^{-1}_D] \wedge \delta A \right)$$

$$= \pm \int_M 2 \varepsilon^{ABCD} \, \mathrm{tr}\left(d_D e^{-1}_C \wedge e^{-1}_D \wedge \delta A \right).$$

$\delta S = 0$ thus implies $\varepsilon^{ABCD} d_D e^{-1}_C \wedge e^{-1}_D = 0$. Since $e^{-1} \neq 0 \Rightarrow d_D e^{-1} = 0$.

The variation w.r.t e yields:

$$\delta S = \int_M \delta \, \mathrm{tr}\left(F \wedge * (e^{-1} \wedge e^{-1}) \right) = \int_M \mathrm{tr}\left(F \wedge (\delta e^{-1}_C \wedge e^{-1}_D + e^{-1}_C \wedge \delta e^{-1}_D) \right) \varepsilon^{ABCD}$$

$$= 2 \int_M \mathrm{tr}\left(F \wedge e^{-1}_C \wedge \delta e^{-1}_D \right) \varepsilon^{ABCD}$$

$$\delta S = 0 \Rightarrow F_{AB} \wedge e^{-1}_C \varepsilon^{ABCD} = 0$$

$$\rightsquigarrow \tilde{R}_{\alpha\beta} - \tfrac{1}{2} \tilde{R} g_{\alpha\beta} = 0.$$

Both, they imply the standard Einstein equation.

Exercise 43:

This kind of seperation can be done because we have the diffeomorphism $\phi : M \to \mathbb{R} \times S$. Having the coordinates (t,x,y,z) on $\mathbb{R} \times S$, we can get the corresponding coordinates on M through the pullback $x^0 = \phi^* t, \ldots$. We can now set arbitrarily $x^0 = \tau$. To get the vector fields $\partial_1, \partial_2, \partial_3$ that are tangent to Σ, we can just choose the tangent components of the respective pullbacks:

$$\partial_1 = \partial_x + g(\partial_x, n) n \quad , \quad \partial_2 = \partial_y + g(\partial_y, n) n \quad , \quad \partial_3 = \partial_z + g(\partial_z, n) n .$$

With this choice $\partial_1, \partial_2, \partial_3 \in T_p\Sigma$ and form a basis since $\partial_x, \partial_y, \partial_z$ form a basis in S. It is also clear from dimensional reasons: One can choose three linearly-independent vectors $\partial_1, \partial_2, \partial_3$ in the three-dimensional submanifold $T_p\Sigma$ and one vector ∂_τ, s.t. they span whole T_pM.

Exercise 44:

$$\mathcal{G}^0_0 = R^{0\alpha}{}_{0\alpha} - \tfrac{1}{2} R^{\alpha\beta}{}_{\alpha\beta} = R^{00}{}_{00} + R^{01}{}_{01} + R^{02}{}_{02} + R^{03}{}_{03} - \tfrac{1}{2}\left(R^{0\beta}{}_{0\beta} + R^{1\beta}{}_{1\beta} + R^{2\beta}{}_{2\beta} + R^{3\beta}{}_{3\beta} \right)$$

$$= \tfrac{1}{2} R^{00}{}_{00} + \cancel{\tfrac{1}{2} R^{01}{}_{01}} + \cancel{\tfrac{1}{2} R^{02}{}_{02}} + \cancel{\tfrac{1}{2} R^{03}{}_{03}} - \cancel{\tfrac{1}{2} R^{10}{}_{10}} - \tfrac{1}{2} R^{12}{}_{12} - \tfrac{1}{2} R^{13}{}_{13} - \cancel{\tfrac{1}{2} R^{20}{}_{20}} - \tfrac{1}{2} R^{21}{}_{21} - \tfrac{1}{2} R^{23}{}_{23} - \cancel{\tfrac{1}{2} R^{30}{}_{30}} - \tfrac{1}{2} R^{31}{}_{31} - \tfrac{1}{2} R^{32}{}_{32}$$

$$= - \left(R^{12}{}_{12} + R^{13}{}_{13} + R^{23}{}_{23} \right).$$

Exercise 45:

$$\tfrac{1}{2} {}^3R = \tfrac{1}{2} {}^3R^{ij}{}_{ij} = \tfrac{1}{2}\left(2 \, {}^3R^{12}{}_{12} + 2 \, {}^3R^{23}{}_{23} + 2 \, {}^3R^{31}{}_{31} \right) = {}^3R^{12}{}_{12} + {}^3R^{23}{}_{23} + {}^3R^{31}{}_{31} .$$

$$- \tfrac{1}{2}\left[(K^i_i)^2 - K^i_j K^j_i \right] = - \tfrac{1}{2}\left[- K^1_2 K^2_1 - K^1_3 K^3_1 - K^2_1 K^1_2 - K^2_3 K^3_2 - K^3_1 K^1_3 - K^3_2 K^2_3 \right] - K^1_1 K^2_2 - K^1_1 K^3_3 - K^2_2 K^3_3$$

Exercise 46:

It seems like there are some sign-mistakes inbetween Ex. 45 and Ex. 46 and the results may differ from equations in the literatur. Also the equations in Ex. 46 seem to have the wrong sign. Using the results $\mathcal{G}^0_0 = -\frac{1}{2}\left({}^3R + (K^i_i)^2 - K_{ij}K^{ij}\right)$, $\mathcal{G}^0_i = {}^3\nabla_j K^j_i - {}^3\nabla_i K^j_j$ from the book, it is easiest to show the given relations by going to the coordinate system where $n = \partial_0$, thus the lapse $N = n^0 = 1$. Since the contractions are Lorentz-scalars/vectors, they are the same in every coordinate system.

- $\mathcal{G}_{\mu\nu} n^\mu n^\nu = \mathcal{G}_{00} n^0 n^0 = -\mathcal{G}^0_0 = \frac{1}{2}\left({}^3R + \mathrm{tr}(K)^2 - \mathrm{tr}(K^2)\right)$.
- $\mathcal{G}_{\mu i} n^\mu = \mathcal{G}_{0i} n^0 = -\mathcal{G}^0_i = {}^3\nabla_i K^j_j - {}^3\nabla_j K^j_i$.

For general lapse and shift, the Gauss equation reads:

- $R^0{}_{ijk} = ({}^3\nabla_i K_{jk} - {}^3\nabla_j K_{ik}) n^0$.
- $R^m{}_{ijk} = ({}^3\nabla_i K_{jk} - {}^3\nabla_j K_{ik}) n^m + ({}^3R^m{}_{ijk} + K_{jk}K^m_i - K_{ik}K^m_j)$
  $= R^0{}_{ijk}\dfrac{n^m}{n^0} + {}^3R^m{}_{ijk} + K_{jk}K^m_i - K_{ik}K^m_j$.

And the above is the Codazzi equation. With $n = \frac{1}{N}(\partial_0 - \vec{N})$ it is $n^0 = \frac{1}{N}$ and $n^m = \dfrac{N^m}{N}$.

Exercise 47:

We have to check that
1) $\{f,g\} = -\{g,f\}$ $\forall f,g \in C^\infty(M)$.
2) $\{f, \alpha g + \beta h\} = \alpha\{f,g\} + \beta\{f,h\}$ $\forall f,g,h \in C^\infty(M)$, $\forall \alpha,\beta \in \mathbb{R}$.
3) $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ $\forall f,g,h \in C^\infty(M)$.

1) $\{f,g\} = \dfrac{\partial f}{\partial p_i}\dfrac{\partial g}{\partial q^i} - \dfrac{\partial f}{\partial q^i}\dfrac{\partial g}{\partial p_i} = -\left(\dfrac{\partial f}{\partial q^i}\dfrac{\partial g}{\partial p_i} - \dfrac{\partial f}{\partial p_i}\dfrac{\partial g}{\partial q^i}\right) = -\{g,f\}$.

2) $\{f, \alpha g + \beta h\} = \dfrac{\partial f}{\partial p_i}\dfrac{\partial(\alpha g + \beta h)}{\partial q^i} - \dfrac{\partial f}{\partial q^i}\dfrac{\partial(\alpha g + \beta h)}{\partial p_i} = \alpha\dfrac{\partial f}{\partial p_i}\dfrac{\partial g}{\partial q^i} + \beta\dfrac{\partial f}{\partial p_i}\dfrac{\partial h}{\partial q^i} - \alpha\dfrac{\partial f}{\partial q^i}\dfrac{\partial g}{\partial p_i} - \beta\dfrac{\partial f}{\partial q^i}\dfrac{\partial h}{\partial p_i}$
$= \alpha\{f,g\} + \beta\{f,h\}$.

3) $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}$

$$= \frac{\partial f}{\partial p_i}\frac{\partial\{g,h\}}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial\{g,h\}}{\partial p_i} + \frac{\partial g}{\partial p_i}\frac{\partial\{h,f\}}{\partial q^i} - \frac{\partial g}{\partial q^i}\frac{\partial\{h,f\}}{\partial p_i} + \frac{\partial h}{\partial p_i}\frac{\partial\{f,g\}}{\partial q^i} - \frac{\partial h}{\partial q^i}\frac{\partial\{f,g\}}{\partial p_i}$$

$$= \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i}\left(\frac{\partial g}{\partial p_j}\frac{\partial h}{\partial q^j} - \frac{\partial g}{\partial q^j}\frac{\partial h}{\partial p_j}\right) - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}\left(\frac{\partial g}{\partial p_j}\frac{\partial h}{\partial q^j} - \frac{\partial g}{\partial q^j}\frac{\partial h}{\partial p_j}\right) + \frac{\partial g}{\partial p_i}\frac{\partial}{\partial q^i}\left(\frac{\partial h}{\partial p_j}\frac{\partial f}{\partial q^j} - \frac{\partial h}{\partial q^j}\frac{\partial f}{\partial p_j}\right) - \frac{\partial g}{\partial q^i}\frac{\partial}{\partial p_i}\left(\frac{\partial h}{\partial p_j}\frac{\partial f}{\partial q^j} - \frac{\partial h}{\partial q^j}\frac{\partial f}{\partial p_j}\right)$$

$$+ \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i}\left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j}\right) - \frac{\partial h}{\partial q^i}\frac{\partial}{\partial p^i}\left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j}\right)$$

$$= \frac{\partial f}{\partial p_i}\frac{\partial^2 g}{\partial q^i\partial p_j}\frac{\partial h}{\partial q^j} + \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial q^i\partial q^j} - \frac{\partial f}{\partial p_i}\frac{\partial^2 g}{\partial q^i\partial q^j}\frac{\partial h}{\partial p_j} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^j}\frac{\partial^2 h}{\partial q^i\partial p_j} - \frac{\partial f}{\partial q^i}\frac{\partial^2 g}{\partial p_i\partial p_j}\frac{\partial h}{\partial q^j} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial p_i\partial q^j} + \frac{\partial f}{\partial q^i}\frac{\partial^2 g}{\partial p_i\partial q^j}\frac{\partial h}{\partial p_j}$$

$$+ \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial q^j}\frac{\partial^2 h}{\partial p_i\partial p_j} + \frac{\partial g}{\partial p_i}\frac{\partial^2 h}{\partial q^i\partial p_j}\frac{\partial f}{\partial q^j} + \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial q^i\partial q^j} - \frac{\partial g}{\partial p_i}\frac{\partial^2 h}{\partial q^i\partial q^j}\frac{\partial f}{\partial p_j} - \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q^j}\frac{\partial^2 f}{\partial q^i\partial p_j} - \frac{\partial g}{\partial q^i}\frac{\partial^2 h}{\partial p_i\partial p_j}\frac{\partial f}{\partial q^j} - \frac{\partial g}{\partial q^i}\frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial p_i\partial q^j}$$

$$+ \frac{\partial g}{\partial q^i}\frac{\partial^2 h}{\partial p_i\partial q^j}\frac{\partial f}{\partial p_j} + \frac{\partial g}{\partial q^i}\frac{\partial h}{\partial q^j}\frac{\partial^2 f}{\partial p_i\partial p_j} + \frac{\partial h}{\partial p_i}\frac{\partial^2 f}{\partial q^i\partial p_j}\frac{\partial g}{\partial q^j} + \frac{\partial h}{\partial p_i}\frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial q^i\partial q^j} - \frac{\partial h}{\partial p_i}\frac{\partial^2 f}{\partial q^i\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial h}{\partial p_i}\frac{\partial f}{\partial q^j}\frac{\partial^2 g}{\partial q^i\partial p_j}$$

$$- \frac{\partial h}{\partial q^i}\frac{\partial^2 f}{\partial p_i\partial p_j}\frac{\partial g}{\partial q^j} - \frac{\partial h}{\partial q^i}\frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial p^i q^j} + \frac{\partial h}{\partial q^i}\frac{\partial^2 f}{\partial p_i\partial q^j}\frac{\partial g}{\partial p_j} + \frac{\partial h}{\partial q^i}\frac{\partial f}{\partial q^j}\frac{\partial^2 g}{\partial p_i\partial p_j} = 0.$$

The Leibniz law can be also easily checked :

$$\{f,gh\} = \frac{\partial f}{\partial p_i}\frac{\partial(gh)}{\partial q^i} - \frac{\partial f}{q^i}\frac{\partial(gh)}{\partial p_i}$$

$$= \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}h + \frac{\partial f}{\partial p_i}g\frac{\partial h}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}h - \frac{\partial f}{\partial q^i}g\frac{\partial h}{\partial p_i}$$

$$= \{f,g\}h + g\{f,h\} .$$

Exercise 48 :

- $[\hat{H},\hat{p}_j] = \frac{1}{2m}[\hat{p}_k\hat{p}^k,\hat{p}_j] = 0$, since $[\hat{p}_k,\hat{p}_j] = 0$.
- $[\hat{H},\hat{q}^j] = \frac{1}{2m}[\hat{p}_k\hat{p}^k,\hat{q}^j] = \frac{1}{2m}\left(\hat{p}_k[\hat{p}^k,\hat{q}^j] + \underbrace{[\hat{p}_k,\hat{q}^j]}_{=-i\delta^j_k}\hat{p}^k\right) = -\frac{i}{m}\hat{p}^j$.

The Poisson brackets read :

- $\{H,p_j\} = \frac{1}{2m}\left(\frac{\partial(p^2)}{\partial p_i}\underbrace{\frac{\partial p_j}{\partial q^i}}_{=0} - \underbrace{\frac{\partial(p^2)}{\partial q^i}}_{=0}\frac{\partial p_j}{\partial p_i}\right) = 0$.
- $\{H,q^j\} = \frac{1}{2m}\left(\frac{\partial(p^2)}{\partial p_i}\frac{\partial q^j}{\partial q^i} - \underbrace{\frac{\partial(p^2)}{\partial q^i}\frac{\partial q^j}{\partial p_i}}_{=0}\right) = \frac{1}{2m}2p^i\delta^j_i = \frac{1}{m}p^j$.

They fullfill :

$$\{f,g\} = k \longrightarrow [\hat{f},\hat{g}] = -i\hat{k} .$$

Exercise 49 :

We have $p^{ij} = q^{1/2}(K^{ij} - tr(K)q^{ij})$, so

$$C = -{}^3R + q^{-1}\left(tr(p^2) - \tfrac{1}{2}tr(p)^2\right)$$

$$= -{}^3R + q^{-1}\left[tr\left(q(K^{ij} - tr(K)q^{ij})^2\right) - \tfrac{1}{2}tr\left(q^{1/2}(K^{ij} - tr(K)q^{ij})\right)^2\right]$$

$$= -{}^3R + \left[tr(K^2 - tr(K)Kq - tr(K)qK + tr(K)^2q^2) - \tfrac{1}{2}[tr(K) - tr(K)tr(q)]^2\right]$$

$$= -{}^3R + tr(K^2) - 2tr(K)tr(Kq) + tr(K)^2tr(q^2) - \tfrac{1}{2}tr(K)^2 + tr(K)^2tr(q) - \tfrac{1}{2}tr(K)^2tr(q)^2$$

$$= -{}^3R + tr(K^2) - \tfrac{1}{2}tr(K)^2 - 2tr(K)^2 + 3tr(K)^2 + 3tr(K)^2 - \tfrac{1}{2}tr(K)^2\cdot 9$$

$$= -{}^3R + tr(K^2) - tr(K)^2$$

$$= -2\,g_{\mu\nu}n^\mu n^\nu .$$

We have used that $tr(q^{ij}) = 3$ and $tr(Kq) = tr(K^{ij}q_{jk}) = tr(K^i_k) = tr(K)$.

Note : In order to have the correct sign here, we have to take our solution and not the one in the book. It has to be a sign error somewhere .

Finally :

$$C_i = -2\,{}^3\nabla^j(q^{-1/2}p_{ij})$$

$$= -2\,{}^3\nabla^j(K_{ij} - tr(K)q_{ij})$$

$$= -2\left[{}^3\nabla^jK_{ij} - tr({}^3\nabla^jK)q_{ij} - tr(K)\underbrace{{}^3\nabla^jq_{ij}}_{=0}\right]$$

$$= -2\left[{}^3\nabla^jK_{ij} - {}^3\nabla_iK^j_j\right] .$$

$$= -2\,g_{\mu i}n^\mu .$$

Exercise 50 :

In local coordinates we have

$$(*F)^{IJ}_{\alpha\beta} = \frac{1}{2}\varepsilon^{IJ}{}_{KL} F^{KL}_{\alpha\beta} = \frac{1}{2}\varepsilon^{IJ}{}_{KL}\left[\partial_\alpha A^{KL}_\beta - \partial_\beta A^{KL}_\alpha + [A_\alpha, A_\beta]^{KL}\right]$$
$$= \partial_\alpha(*A^{IJ}_\beta) - \partial_\beta(*A^{IJ}_\alpha) + *[A_\alpha, A_\beta]^{IJ}$$
$$= i\partial_\alpha A^{IJ}_\beta - i\partial_\beta A^{IJ}_\alpha + i[A_\alpha, A_\beta]^{IJ}$$
$$= i F^{IJ}_{\alpha\beta}.$$

We have used that A is self-dual $\left((*A)^{IJ}_\alpha = \frac{1}{2}\varepsilon^{IJ}{}_{KL}A^{KL}_\alpha\right)$ and that the commutator of two self-dual matrices is again self-dual.

Exercise 51 :

The complexification of a real Lie algebra $\mathfrak{g}$ is a complex Lie algebra.
The vector space $\mathfrak{g}\otimes\mathbb{C}$ equipped with the map $[\cdot,\cdot]: \mathfrak{g}\otimes\mathbb{C}\times\mathfrak{g}\otimes\mathbb{C} \to \mathfrak{g}\otimes\mathbb{C}$
makes up a complex Lie algebra, since $(x\otimes\alpha, y\otimes\beta) \mapsto [x,y]\otimes\alpha\beta$

1) $[x\otimes\alpha, y\otimes\beta] = [x,y]\otimes\alpha\beta = -[y,x]\otimes\beta\alpha = [y\otimes\beta, x\otimes\alpha]$.

2) $[x\otimes\alpha, z(y\otimes\beta)+w(c\otimes\beta)] = [x, zy+wc]\otimes\alpha\beta = (z[x,y]+w[x,c])\otimes\alpha\beta$
$= z[x\otimes\alpha, y\otimes\beta] + w[x\otimes\alpha, c\otimes\beta]$.

3) $[x\otimes\alpha,[y\otimes\beta, z\otimes\gamma]] + [y\otimes\beta,[z\otimes\gamma, x\otimes\alpha]] + [z\otimes\gamma,[x\otimes\alpha, y\otimes\beta]]$
$= [x\otimes\alpha, [y,z]\otimes\beta\gamma] + [y\otimes\beta, [z,x]\otimes\gamma\alpha] + [z\otimes\gamma, [x,y]\otimes\alpha\beta]$
$= [x,[y,z]]\otimes\alpha\beta\gamma + [y,[z,x]]\otimes\beta\gamma\alpha + [z,[x,y]]\otimes\gamma\alpha\beta$
$= ([x,[y,z]] + [y,[z,x]] + [z,[x,y]])\otimes\alpha\beta\gamma$
$= 0$.

It is $\mathfrak{g}_\pm \subset \mathfrak{g}\otimes\mathbb{C}$ and it's closed under the map $[\cdot,\cdot]$, s.t. $[\cdot,\cdot]: \mathfrak{g}_\pm\times\mathfrak{g}_\pm \to \mathfrak{g}_\pm$ as one can easily see.

different sign than in book

They are obviously isomorph to $\mathfrak{g}$: $\Phi: \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}_\pm$, $x \mapsto x\otimes 1 \pm x\otimes i$.

Since $\mathfrak{g}\otimes\mathbb{C} = \{x\otimes z \mid x\in\mathfrak{g}, z\in\mathbb{C}\}$, $\mathfrak{g}\otimes\mathbb{C}$ is given as the direct sum of $\mathfrak{g}_\pm$.

Exercise 52 :

The computations are similar to the computations for the Palatini formalism in Chapter III.3.

1. Varying the self-dual connection :

$$\delta S_{SD} = \int_M (\delta^+\tilde{R})\,\mathrm{vol} = \int_M g^{\alpha\beta}(\delta^+\tilde{R}_{\alpha\beta})\,\mathrm{vol}$$

With the formula $\delta^+\tilde{R}_{\alpha\beta} = 2\tilde{\nabla}_{[\alpha}\delta^+\tilde{\Gamma}^\gamma_{\gamma]\beta}$, we can write ${}^+\tilde{\Gamma}^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} + C^\gamma_{\alpha\beta}$ analog to Chap. III.3 with $\delta^+\tilde{\Gamma}^\gamma_{\alpha\beta} = \delta C^\gamma_{\alpha\beta}$.

Analog to the Palatini formalism, the variation of $\delta^+\tilde{R}$ vanishes iff $C^\gamma_{\alpha\beta} = 0 \rightarrow {}^+\tilde{\Gamma}^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}$

2. Varying the frame field e_I^α:

It is $\delta vol = -\frac{1}{2} g_{\alpha\beta} (\delta g^{\alpha\beta}) vol$ with $\delta g^{\alpha\beta} = \delta(\eta^{IJ} e_I^\alpha e_J^\beta) = 2\eta^{IJ} e_J^\beta \delta e_I^\alpha$.

$$\Rightarrow \delta S_{SD} = \int_M [(\delta e_I^\alpha) e_J^\beta {}^+F_{\alpha\beta}^{IJ} + e_I^\alpha (\delta e_J^\beta) {}^+F_{\alpha\beta}^{IJ} - e_I^\alpha e_J^\beta {}^+F_{\alpha\beta}^{IJ} e_\gamma^K (\delta e_K^\gamma)] vol$$

$$= 2\int_M [e_J^\beta {}^+F_{\alpha\beta}^{IJ} - \frac{1}{2} e_\alpha^I e_K^\gamma e_L^\delta {}^+F_{\gamma\delta}^{KL}] (\delta e_I^\alpha) vol$$

$$= 2\int_M [{}^+R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} {}^+R] \eta^{IJ} e_J^\beta (\delta e_I^\alpha) vol.$$

$$\rightsquigarrow {}^+R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} {}^+R = 0.$$

Like in the case of the normal Einstein equation, we see by contracting

$${}^+R^\alpha_\alpha - \frac{1}{2}\delta^\alpha_\alpha {}^+R = 0 \iff -{}^+R = 0 \iff {}^+R = 0.$$

So the vacuum Einstein equation becomes ${}^+R_{\alpha\beta} = 0$, which means that

$$\frac{1}{2}(R^\gamma{}_{\alpha\gamma\beta} - \frac{i}{2}\varepsilon^\gamma{}_{\beta\mu\nu} R^\mu{}_{\alpha\gamma}{}^\nu) = 0 \iff R_{\alpha\beta} = \frac{i}{2}\varepsilon^\gamma{}_{\beta\mu\nu} R^\mu{}_{\alpha\gamma}{}^\nu.$$

Since the Ricci tensor is real-valued, the above equation can be only fullfilled iff $R_{\alpha\beta} = 0$.

Exercise 53:

Working on it ! ;)